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Testing Chaotic Dynamics via Lyapunov Exponents

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ABSTRACT

In this paper, we propose a new test, based on the stability of the largest Lyapunov exponent from different sample sizes, to detect chaotic dynamics in time series. We apply this new test to the simulated data used in the single-blind controlled competition among tests for nonlinearity and chaos generated by Barnett *et al.* (1997), as well as to several chaotic series, both for small and large samples. The results suggest that the new test has a high power against different stochastic alternatives (both linear and nonlinear), and also performs well in small samples.

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1. Introduction

In a dissipative dynamical system, the existence of a positive Lyapunov exponent is usually taken as an indication on the chaotic character of the system. Lyapunov exponents provide information on the intrinsic instability of the trajectories of the system, and are computed as the average rate of exponential convergence or divergence of nearby trajectories in the phase space.

In recent years, there has been a burgeoning literature on the calculation of Lyapunov exponents for an unknown dynamical system reconstructed from a single time series. The seminal paper of Wolf *et al*. (1985) provides an algorithm to compute Lyapunov exponents in empirical applications, but this is sensitive to both the number of observations and the degree of noise in the data. More recently, however, some authors have proposed new methods for estimating Lyapunov exponents, showing a good performance even for small samples [see, among others, Dechert and Gençay (1992), Abarbanel *et al*. (1991, 1992), and Rosenstein *et al*. (1993)].

There are many papers using Lyapunov exponents to detect chaotic dynamics in financial time series, especially in exchange rate series. Earlier examples of research in this area include Bajo-Rubio *et al*. (1992) and Dechert and Gençay (1992), where Lyapunov exponents are used to distinguish between linear, deterministic processes (with negative Lyapunov exponents) and nonlinear, chaotic deterministic processes (where the largest Lyapunov exponent is positive). These and other papers have been criticised for the absence of a distributional theory providing a statistical framework for hypothesis testing using the calculated Lyapunov exponents. However, Gençay (1996) presents a methodology to compute the empirical distributions of Lyapunov exponents using a blockwise bootstrap technique. This methodology provides a formal test of the hypothesis that the largest Lyapunov exponent equals some hypothesised value, and can be used to test for chaotic dynamics. The test proposed by Gençay (1996) is particularly useful in those cases where the largest Lyapunov exponent is positive, but very close to zero. More recently, Bask and Gençay (1998) utilise the same statistical framework to provide a test for the presence of a positive Lyapunov exponent in an observed time series. The numerical examples show that both Gençay (1996) and Bask and Gençay (1998) test statistics behave well in small samples. Finally, Bask (1998), using the test suggested by Bask and Gençay's (1998) test, finds evidence that some exchange rates can be characterised by deterministic chaos.

Despite the growing interest on the econometric literature aimed to distinguish between non-linear deterministic processes and non-linear stochastic processes, there is still a lot of disagreement and controversy about the available results. A key paper in this area is Barnett *et al*. (1997), where some data series were simulated from different generating models in order to evaluate the behaviour, both for large and small samples, of five highly regarded tests for nonlinearity or chaos. The tests considered in that paper are the Hinich bispectral test (Hinich,1982), the BDS test (Brock *et al*., 1996), the NEGM test (Nychka *et al*., 1992), the White test (White, 1989), and the Kaplan test (Kaplan, 1993). The results about the power function of some of such tests proved to be rather surprising , since none of them had the ability to isolate the origins of the nonlinearity or chaos to be within the structure of the economy.

The aim of this paper is to propose a new test for the presence of chaos, based on the behaviour of the estimated Lyapunov exponents, for different sample sizes. As we shall try to illustrate, while the largest exponent of a chaotic process is invariant with respect to sample size, the largest Lyapunov exponent of a stochastic process is not. Therefore, we suggest testing chaotic dynamics by estimating the empirical distributions of the largest Lyapunov exponents for different subsamples and comparing their means. The proposed new test shows a strong power against stochastic processes, hence providing further refinement over those of Gençay (1996) and Bask and Gençay (1998).

The rest of the paper is organised as follows. Section 2 presents the statistical framework used in the paper. Section 3 discusses the stability of the largest Lyapunov exponent with sample size. Section 4 proposes the new test for distinguishing chaos from random behaviour. Section 5 reports the results of applying our test to several chaotic processes, as well as to the simulated data used in the single-blind controlled competition performed by Barnett *et al*. (1997). Section 6 presents a comparison with Bask and Gençay's (1998) test. Finally, Section 7 provides some concluding remarks.

2. A statistical framework for testing chaotic dynamics via Lyapunov exponents

In order to examine the properties of deterministic dynamical system we make use of ergodic theory, since it provides a statistical framework where different degrees of complexity of attractors and motions can be distinguished [see Eckmann and Ruelle (1985) for a survey]. Furthermore, ergodic theory allows us to describe the time averages of a dynamical system and to consider that transients become irrelevant. Once transients are over, the motion of the dynamical system settles typically near a subset of $\Re^n$, called an *attractor.* In the particular case of dissipative systems, where the phase-space volumes are concentrated by the time evolution, the volume occupied by the attractor is in general very small in relation to the phase space. Even if a system contracts its volume, it does not mean that its length is contracted in all directions: some directions may be stretched and some directions contracted. This implies that, even in a dissipative system, the final motions may be unstable within the attractor. This instability usually manifests itself in *sensitive dependence on initial conditions*, which means an exponential separation of orbits (as time goes on) of points that were initially very close each other on the attractor. In this case, we say that the attractor is a *strange attractor* and that the system is *chaotic*.

Statistical averages can be computed either in terms of time averages or space averages. Let us consider, for simplicity, a discrete dynamical system of dimension n $\vec{x}_{t+1} = \vec{F}(\vec{x}_t)$, where $\vec{F} : \Re^n \longrightarrow \Re^n$ is a vectorial differentiable function. The *time average* of a function j along a (forward) trajectory $\vec{x}_i$ with initial condition $\vec{x}_0$, of a discrete dynamical system is defined by

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N-1} j(\vec{x}_i)$$

In a similar way for a continuous flow f_t, arising from a continuous dynamical system $\frac{d\vec{x}}{dt} = \vec{F}(\vec{x})$ the time average of a function along a (forward) trajectory is

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T j\,(f_t(x))\,dt$$

The time averages often depend on initial conditions. However, when the dynamical system has an attractor, all trajectories have the same statistical properties.

A measure of complexity in chaotic motion may be obtained by analyzing the sensitivity of the dynamical behaviour to initial conditions given by two infinitely close initial states. For chaotic systems nearby points in the phase space separate exponentially with time. Let us illustrate the basic idea by means of a discrete dynamical system of dimension *n*, $\vec{x}_{t+1} = \vec{F}(\vec{x}_t)$. In order to examine the stability of the trajectories of the system, let us consider how the system amplifies a small difference between the initial conditions $\vec{x}_0$ and $\vec{x}'_0$:

$$x_T - x'_T = F^T(x_0) - F^T(x'_0) \cong DF^T(x_0)(x_0 - x'_0)$$

where $F^T(x_0) = F(F(...F(x_0)...))$ denotes the T successive iterations of the dynamical system starting from the initial condition $\vec{x}_0$, and where $D\vec{F}^T(\vec{x}_0)$ is the Jacobian of function $\vec{F}(\vec{x})$.

By the rule of the chain, we have

$$DF^T(x_0) = DF(x_{T-1})DF(x_{T-2}).........DF(x_0)$$

In this context, the Lyapunov exponents are defined as follows (Guckenheimer and Holmes, 1990): Let us consider the family of subspaces $V_i^{(1)} \supset V_i^{(2)} \supset ..... \supset V_i^{(n)}$ in the tangent space at $F^i(x)$ and the numbers $l_1 \geq l_2 \geq ..... \geq l_n$ with the properties that:

(1) $D\vec{F}(V_i^{(j)}) = V_{i+1}^{(j)}$

(2) $\dim V_i^{(j)} = n + 1 - j$

(3) $\lim\limits_{T\to\infty} \frac{1}{T} \ln || \sqrt{(DF^T)^* \cdot (DF^T)}(x_0) || = l_j$ for all $x_0 \in V_0^{(j)} - V_0^{(j+1)}$, where $(DF^T)^*$ is the transpose of DF^T

Then, the real numbers l_j are called the *Lyapunov exponents* of $\vec{F}$ at $\vec{x}_0$. Lyapunov exponents offer information on how orbits on the attractor move apart (or together) given the evolution of dynamics. One can also define them by the rate of stretching or shrinking of line segments, areas, and various dimensional subvolumes in the phase space. Line segments grow or shrink as e^{tl_1}, areas as $e^{t(l_1+l_2)}$ and so forth. If one or more of the Lyapunov exponents are positive, then we have *chaos* in the motion of the system. The sum of the Lyapunov exponents is negative ($l_1 + l_2 + ... + l_n < 0$) for dissipative systems [see Abarbanel (1996)].

The possibility of obtaining, in a deterministic dynamical system, Lyapunov exponents that are representative of short-run divergences in trajectories with very closed initial points is based on Oseledec**'s** (1968) *multiplicative ergodic theorem*. If we assume that there exists an ergodic measure of the system, this theorem justifies the use of arbitrary phase space directions when calculating the largest Lyapunov exponent. The Lyapunov exponents have then a mean in a global sense, allowing to characterize the complexity of a deterministic dynamical system of dimension n simply by n real numbers.

Oseledec's (1968) multiplicative ergodic theorem states that, under wide general conditions for function $\vec{F}$, the limit of expression (3) does exist for almost all $\vec{x}_0$ (with respect to the invariant measure m) and is independent of the initial condition x_0 considered (except for a set of null measure). Therefore, the multiplicative ergodic theorem implies that the Lyapunov exponents are invariant numbers representing "globally" the complexity of the dynamical system under study, independently of the initial condition considered.

Oseledec theorem is based on the ergodic theory of deterministic dynamical systems and justifies the use of arbitrary phase space directions when calculating the largest Lyapunov exponents. Nevertheless, as both Whang and Linton (1999) and Tong (1990) point out, Lyapunov exponents can be interpreted within the standard non-linear time series framework as a measure of local stability and is of interest even outside from any direct connection with deterministic chaos.

Within the theory of dynamical systems, a chaotic system is characterised by globally bounded trajectories in the phase space with a positive largest Lyapunov exponent, while, in theory, a white noise process has an infinite largest Lyapunov exponent (see Schuster, 1988).

Nevertheless, in practical implementations, using finite time series, any standard algorithm for calculating the largest Lyapunov exponent will find a finite, positive value for this exponent for a random process. Therefore, the largest Lyapunov exponent on its own is not able to distinguish between a chaotic, non-linear deterministic process and a random process. This problem is especially relevant in financial time series, where non-linear stochastic processes, such as GARCH processes, are usually postulated as alternative models to the chaotic behaviour [see, e. g., Hsieh (1991)].

Gençay (1996) proposed a statistical framework for testing chaotic dynamics using a moving blocks bootstrap procedure.

Consider a sequence $\{X_1, X_2, ..., X_N\}$ of weakly dependent stationary random variables, being $\{x_1, x_2, ..., x_N\}$ a time series realisation of such a stochastic process. According to Künsch (1989) and Liu and Singh (1992), the distribution of certain estimators of interest can be consistently constructed by applying moving blockwise bootstrap. Let $B_t^d = \{x_t, x_{t-1}, ..., x_{t+d-1}\}$ denote a moving block of d consecutive observations. For a time series of N elements, we can form a set $\{B_1^d, ..., B_{N-d+1}^d\}$ of blocks with length d. Let us consider k=int(N/d) [where *int()* denotes the integer part], by resampling with replacement of k blocks denoted by $\{B_{i_1}^d, ..., B_{i_k}^d\}$, we will form the bootstrap sample.

In order to obtain the sample distribution of the largest Lyapunov exponent l_{max}, we will repeat this procedure to construct a sequence of sub-families of k blocks taken with replacement from the family of d-dimensional blocks $\{B_1^d, ..., B_{N-d+1}^d\}$, that can be generated with the time series $\{x_1, x_2, ..., x_N\}$.

For each subfamily of k blocks, we can apply some standard procedure to compute for the largest Lyapunov exponent $\tilde{l}_{max}$ by taking the pairs of nearest neighbours from each subfamily of blocks. Repeating this process a large number of times, we will obtain the empirical distribution of the largest Lyapunov exponent $\tilde{l}_{max}$.

There are several suitable estimation methods in order to obtain Lyapunov exponents based on kernels, nearest neighbors, splines, local polynomials and neural nets [see Härdle and Linton (1994) for a general discussion]. McCaffrey *et al.* (1992) distinguish two classes of methods for estimating the largest Lyapunov exponent l_{max}: (i) Direct methods like Wolf *et al.*'s (1985), which assume that the initial divergence $(\vec{x}_0 - \vec{x}_0')$ grows at the exponential rate given by l_{max}; and (ii) Jacobian methods, where data are used to estimate the Jacobians, with l_{max} computed from the estimated Jacobians, like those proposed by MacCaffrey *et al.* (1992) or Gençay (1996). On the other hand, Gençay and Dechert (1992), Gençay and Dechert (1996) and Dechert and Gençay (2000), have studied the topological invariance of the Lyapunov Exponent estimator from observer dynamics.

As Ziehmann *et al.* (1999) pointed out, a bootstrap algorithm must be used with caution if Lyapunov exponents estimates rely on the product of matrices because matrix multiplication does not commute, except in one dimension.. In order to avoid such complications with the product of Jacobians along the trajectory, we use a simple direct method for estimating the largest Lyapunov exponent l_{max} of a time series proposed by Rosenstein *et al.* (1993). Given that the divergence between the nearest neighbours takes place at a rate approximated by the largest Lyapunov exponent, Rosenstein *et al.* suggest to choose a pair of neighbours as nearby initial conditions for different trajectories, and to estimate l_{max} by averaging exponential divergences of initially close state-space trajectories.

In the method proposed by Rosenstein *et al.* (1993), there are two key parameters to estimate the largest Lyapunov exponent: the embedding dimension (that will be the moving-block length for the moving blocks bootstrap procedure) and the number of discrete-time steps allowed for divergence between nearest neighbors in the phase space. As shown in Rosenstein *et al.* (1993), the value of the largest Lyapunov exponent can be biased with these two parameters.

3. Stability of largest Lyapunov exponents with the sample size for chaotic processes

From a theoretical point of view, the reason for the stability of the largest Lyapunov exponent with respect to the sample size can be found in Oseledec's (1968) theorem. This theorem allows us to affirm that, for a large enough sample size, these exponents will converge to some stable values associated with the complexity of the attractor.

This theorem assures, for chaotic time series, the possibility of making short-run forecasts based on the reconstructed phase space. The Lyapunov exponents are nothing but a measure (in exponential scale) of the mean forecast errors using the nearest neighbour points in the phase space. However, when analysing a time series generated by a non-deterministic stochastic process, nothing guarantees the stability of the Lyapunov exponents. Oseledec's (1968) theorem only affects deterministic processes via ergodic theory. For a stochastic process, as the number of observations increases, the variability of the largest Lyapunov exponent will be greater and, therefore, the largest Lyapunov exponent will also increase without limit with the sample size.

As we shall see, our simulations show an essential difference between chaotic and stochastic processes via Lyapunov exponents. If we want to reconstruct trajectories of a time series in a phase space that are sampled from a stochastic process, there is not guarantee of convergence in any algorithm towards the largest Lyapunov exponent, because the Lyapunov exponents are not necessarily stable and independent of the initial conditions and sample size. For stochastic processes, the algorithm is only able to estimate *local Lyapunov exponents*. Local Lyapunov exponents are a measure of local stability of the process and may be highly dependent on the sample size and the initial condition considered.

Our simulations are based on different stochastic and chaotic processes. First of all, and following Barnett et al. (1997), let us consider samples of size 380 and 2000 observations of the following five models:

(i) A fully deterministic, chaotic Feigenbaum recursion of the form:

$$y_t = 3.57\, y_{t-1}(1 - y_{t-1}),$$

where the initial condition was set at $y_0 = 0.7$[1].

[1] The Feigenbaum series proposed in Barnett *et al.* (1997) [i.e., $y_t = c\, y_{t-1}(1 - y_{t-1}), c = 3.57, y_0 = 0.7$] is really special as can be seen in Fernández-Rodríguez et al. (2000). The problem is that the parameter $c = 3.57$ of this map is too close to $c_\infty = 3.569946..$, the value of the parameter where the period 2^n $(n \to \infty)$ cycle first occurs [see Jackson, 1989]. For $c < c_\infty$ we have 2^n cycles and for $c_\infty \le c \le 4$ the map displays a rich variety of behaviours [see Jackson, 1989]. For $c > c_\infty$, except for the narrow bands where the solutions would oscillate again according to an *n*-cycle (e.g. $n = 3$ for $3.83 < c < 3.86$), there is an infinite number of possible values for y_t that never repeats itself.

For $c = 3.57 \cong c_\infty \cong 3.569946$ used by Barnett *et al.* (1997), the sequence generated by the Feigenbaum map is much less "regular" than a sequence with a finite period of repetition. Nevertheless, the c_∞ sequence has an important difference with true chaotic behaviour. The reason is that the c_∞ sequence is still marginally predictable in the sense that if two initial values are close enough to each other, the two sequences generated by Feigenbaum map, *(ii)* A GARCH process of the following form:

$$y_t = h_t^{1/2} u_t ,$$

where h_t is defined by

$$h_t = 1 + 0.1 y_{t-1}^2 + 0.8 h_{t-1} ,$$

with $h_0 = 1$ and $y_0 = 0$.

(iii) A nonlinear moving average (NLMA) process:

$$y_t = u_t + 0.8 u_{t-1} u_{t-2} .$$

(iv) An ARCH process of the following form:

$$y_t = (1 + 0.5 y_{t-1}^2)^{1/2} u_t ,$$

with the value of the initial observations is set at $y_0 = 0$, and

(v) An ARMA model of the form:

$$y_t = 0.8 y_{t-1} + 0.15 y_{t-2} + u_t + 0.3 u_{t-1} ,$$

with $y_0 = 1$ and $y_1 = 0.7$.

With the four stochastic models, the white noise disturbances, u_t, are sampled independently from a standard normal distribution. Note that of the five generating models, only model (i) is chaotic.

In order to provide more and stronger evidence supporting our claim on the observed invariance property of the largest Lyapunov exponent holds for all chaotic processes, we also consider the Hénon map and the Lorenz attractor.

(vi) The Hénon (1976) map is described by the following system:

$$\begin{aligned} x_{t+1} &= 1 - 1.4 x_t^2 + y_t \\ y_{t+1} &= 0.3 x_t \end{aligned}$$

with the initial points $x_0 = 0.5$ and $y_0 = 0.2$.

(vii) The well-known Lorenz (1963) attractor is the three-dimensional continuous-time system:

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= x(28 - z) - y \\ \dot{z} &= xy - \frac{8}{3} z \end{aligned} .$$

for these two initial conditions, will be very closed to each other even after a very long time. This is so because, at $c = c_\infty$, the infinite cycle is stable.

Lorenz´s system was solved using a straightforward fourth-order Runge-Kutta method. Considering the average mutual information $I(T)$ for the signal $x(t)$, the minimum of this function is at T=10; following Abarbanel (1996), a time lag $t = 10$ was used in order to obtain a series $x(t_0 + nt),\quad n = 1,\ldots,2000$ as is usual for the phase reconstruction.

We calculated the largest Lyapunov exponent applying the algorithm proposed by Rosenstein *et al.* (1993) to the time series generated by these models for each sample size between 100 and 2000 taken by groups of a hundred (i. e., 100, 200, 300,...2000). We also consider different embedding dimensions from $d=2$ *to* 6. Finally, regarding the number of discrete-time steps allowed for divergence between nearest neighbours, we take $i = 2$.

Figures 1 to 5 display the results of estimating the largest Lyapunov exponents $\hat{l}_{max}$ for the simulated data series used in Barnett *et al*. (1997) competition and for the two new chaotic series (Hénon map and Lorenz attractor) for different sample sizes.

[Figures 1 to 5, Appendix C]

As can be seen, as the embedding dimension increases, the estimated $\hat{l}_{max}$ from this algorithm is gradually downward-biased.

Given the evidence presented in Figures 1 to 5, the existence of a positive largest Lyapunov exponent does not allow to infer the presence of chaos in a given time series. However, these figures show an interesting and essential difference between deterministic and stochastic processes. While the largest Lyapunov exponent, in the case of the deterministic models, stabilises (in some cases even decreases) as the sample size increases, for all the stochastic processes, the largest Lyapunov exponent increases with the sample size. This behaviour remembers the well-known process of saturation of the correlation dimension, in a chaotic time series, when the embedding dimension increases. As a matter of fact, this is the base of the test proposed by Grassberger and Procaccia (1983) to detect deterministic chaos.

4. A new test for distinguish chaos from random behaviour via Lyapunov exponents

In this section, we propose a new test, based on the stability of the largest Lyapunov exponent from different sample sizes, to detect chaotic dynamics in time series. As we will see, this new test has a high power against different stochastic alternatives, both linear and nonlinear.

This new test has a deterministic process as the null hypothesis, while the alternative hypothesis is that of a stochastic process (i. e., high-dimensional chaos), since randomness can be viewed as infinite-dimensional chaos.

Let be a time series of length *N*, $\{x_1, x_2, \ldots\ldots, x_N\}$. Let us divide the time series into different subsamples $\{x_1, x_2, \ldots x_{T_1}, \ldots., x_{T_2}, \ldots., x_{T_{r-1}}, \ldots., x_{T_r} = x_N\}$, and consider an empirical distribution of the largest Lyapunov exponent from 100 moving block bootstrap of such time series for the different subsamples $\{x_1, x_2, \ldots\ldots, x_{T_i}\}$, for $i=1,\ldots.,r$. Let $\langle l_{max}(T_i)\rangle$ be the mean of distributions of the 100 largest Lyapunov exponents computed from those sample sizes. Given that we have shown that the largest Lyapunov exponent stabilises when increasing the sample size in a deterministic process, but it increases with the sample size in a stochastic process, we propose using $\langle l_{max}(T_i)\rangle$ to test for the stability of the largest Lyapunov exponent.

The equality of means may be tested recalling the traditional econometric test of lineal independence between the mean of largest Lyapunov exponents $\langle l_{max}(T)\rangle$, in every sample size, and the sample size T. To that end, we have performed a linear regression of

$$\langle l_{max}(T)\rangle = a_0 + a_1 T + e_T \quad for\, T = T_1, \ldots\ldots, T_r = N\,, \tag{1}$$

so that the estimated parameter $\hat{a}_1$ can be used to test if the largest Lyapunov exponent does not increase with sample size, implying a underlying deterministic process.

The null hypothesis H_0 and the alternative hypothesis H_1 are formulated as follows:

$H_0 : a_1 \leq 0$ *(deterministic process)*

$H_1 : a_1 > 0$ *(stochastic process)*

5 Applications

In this section we test for deterministic chaos using the simulated data from the seven models presented in the previous section. In all cases, the largest Lyapunov exponents were estimated using the algorithm proposed in Rosenstein *et al*. (1993).

Following Barnett *et al*. (1997), we compute our tests twice: for small samples of 380 observations and for large samples of 2000 observations. For the 380 observations case, the subsample sizes are as follows:

$$T_1 = 190, T_2 = 211, T_3 = 232, T_4 = 253, T_5 = 274,$$
$$T_6 = 295, T_7 = 316, T_8 = 337, T_9 = 358, T_{10} = 380.$$

For the 2000 observations case, the subsample sizes are as follows:

$$T_1 = 1000, T_2 = 1100, T_3 = 1200, T_4 = 1300, T_5 = 1400, T_6 = 1500,$$
$$T_7 = 1600, T_8 = 1700, T_9 = 1800, T_{10} = 1900, T_{11} = 2000.$$

In order to implement our test, we compute the largest Lyapunov exponent for small subsample sizes and large subsample sizes for moving-block lengths *d* between 2 and 6.

Tables 1 to 7 [Appendix A] show the results of our regression thesing for the stability of the mean largest Lyapunov exponent $\langle l_{max}(T) \rangle$ (for a sample of 100 largest Lyapunov exponents estimated by bootstraping) when the sample size T increases.

As can be seen, if the 1% marginal significance level is used, for the Feigenbaum (Table 1) and Hénon (Table 6) series the test correctly distinguish deterministic from random behaviour for sample sizes of 380 and 2000 observations. Regarding the Lorenz attractor (Table 7), for T=380, the tests incorrectly reject the null hypothesis for small (d=2) and large (d=6) embedding dimensions, while for T=2000, it only incorrectly rejects the null hypothesis for d=2. Finally, note that the test correctly rejects the null hypothesis for all the stochastic processes (see Tables 2 to 5).

[Tables 1 to 7]

Therefore, our simulation results suggest that our test correctly reject chaos for the GARCH, NLMA, ARCH and ARMA stochastic processes in small and large sample sizes and for all embedding dimensions. On the contrary, our test does not reject chaos in the case of the best-known chaotic processes for moving-block lengths from 3 to 5. According to these results, it appears that a strategy of using d =3 , d=4 or d= 5 could be recommended for practitioners when applying our test.

6. Comparison of the power performance of the new test with Bask and Gençay's (1998)

Bask and Gençay (1998) proposed using a moving-blocks bootstrap procedure to test for the presence of a positive Lyapunov exponent in an observed stochastic time series. The null hypothesis H_0 and the alternative hypothesis H_1 are formulated as follows:

$H_0: \ l_{max} = 0$ *(no chaotic process)*

$H_1: \ l_{max} > 0$ *(chaotic process)*

where l_{max} is the largest Lyapunov exponent.

The test scheme consists of the following steps:

i) Reconstruct the phase space of the time series $\{x_1, x_2, \ldots.., x_N\}$ with a embedding dimension d and estimate the largest Lyapunov exponent $\hat{l}_{max}$ using any existing algorithm. Each d-history of the reconstructed phase space will be considered as a block, obtaining in this way a sequence of blocks $\{ B_1^d, \ldots., B_{N-d+1}^d \}$.

ii) Resample, with replacement, k blocks of the reconstructed phase space, being k=int(N/d). The subfamily of blocks $\{ B_{i_1}^d, \ldots., B_{i_k}^d \}$ constitutes the bootstrap sample.

iii) From this subfamily, estimate the largest Lyapunov exponent $\tilde{l}_{max}$ for the time series under study and calculate $\tilde{l}_{max} - \hat{l}_{max}$.

iv) Repeat steps ii)-iii) a large number of times to construct an empirical distribution of $\tilde{l}_{max} - \hat{l}_{max}$

v) Construct a one-sided 97,5% confidence interval by calculating the critical value as $\hat{l}_{max} - q(97.5\%)$, following from $Pr\{\tilde{l}_{max} - \hat{l}_{max} < q(97.5\%)\} = 0.975$, where $q(97.5\%)$ is the quartile of the distribution in step iv)

vi) If $\hat{l}_{max} - q(97.5\%) > 0$, then the null hypothesis is rejected, which means that the dynamics is chaotic.

Tables 8 to 14 show the results of applying the test proposed by Bask and Gençay (1998) to the simulated data examined in the previous section for moving blocks of different sizes (i. e., d=2 to 6) and for different sample sizes (T=380 and 2000).

[Tables 8 to 14, Appendix B]

As can be seen, this test incorrectly rejects the null the hypothesis, not rejecting the alternative hypothesis $l_{max} > 0$ in any of the stochastic processes considered in this paper (GARCH, NLMA, ARCH and ARMA). On the contrary, the test does correctly reject the null hypothesis, not rejecting the alternative hypothesis for the chaotic processes (Feigenbaum, Hénon and Lorenz).

7. Concluding remarks

Empirical research on detection of chaotic behaviour has expanded rapidly, but results have tended to be rather inconclusive, due to the lack of appropriate testing methods.

The general practice has been to take the existence of a positive Lyapunov exponent as an indication that the system is chaotic. However, this condition is not sufficient for the detection of chaos, and does not help us to distinguish a chaotic process from a stochastic one. Indeed, any standard algorithm for calculating the largest Lyapunov exponent will find a finite, positive value for this exponent, both for chaotic as well as for stochastic processes.

In this paper, we combine the bootstrap statistical framework for hypothesis testing using the computed Lyapunov exponents (Gençay, 1996), with the ergodic theory of deterministic dynamical systems in order to develop a new test to detect chaotic dynamics in time series. The new test is based on the stability of the mean in the distributions of the largest Lyapunov exponent estimated from different sample sizes, which is guaranteed by Oseledec's (1968) theorem. This theorem provides a strong feature of deterministic processes that is not shared by stochastic processes. We show that, while for (linear and nonlinear) stochastic processes the largest Lyapunov exponent increases with the sample size, for chaotic series the largest Lyapunov exponent is invariant when increasing the sample size. We compute the largest Lyapunov exponent using a robust version of the algorithm proposed by Rosenstein *et al.* (1993), considering the mean of divergences between pairs of close trajectories.

We have applied this new test to the simulated data used in the single-blind controlled competition among tests for nonlinearity and chaos generated by Barnett *et al.* (1997), as well as several chaotic series, both for small and large samples (380 and 2000 observations, respectively). The results suggest that the new test has a high discriminatory power against interesting stochastic alternatives, both linear and nonlinear (GARCH, NLMA, ARCH and ARMA).

Indeed, the proposed new test is able to correctly distinguish deterministic from random behaviour for sample sizes of 380 and 2000 observations, and for moving-block lengths from 3 to 5. However, for extreme values of the block size (2 and 6) our test incorrectly rejects the null hypothesis for sample size of 380 observations for one of the three chaotic processes considered. For sample size of 2000 observations our test does not correctly reject the null for all chaotic processes in almost every block size lower than 7, except for the Lorenz series in block size 2.

Finally, our test for stability of largest Lyapunov exponent correctly rejects the null hypothesis of determinism for all the stochastic process (GARCH, NLMA, ARCH and ARMA), regardless the sample size and for all block size considered. On the contrary, our test does not reject chaos in the case of the best-known chaotic processes for block size from 3 to 5.

When comparing the results from our test with those from the competing test proposed by Bask and Gençay (1998), we conclude that the latter cannot correctly distinguish between chaotic and stochastic processes, although it correctly rejects the null hypothesis (of no chaos) for the chaotic processes.

Therefore, the results presented in this paper suggest that our test improves over several tests available in the literature, since it has the ability to distinguish between deterministic or stochastic processes. In addition, our test behaves well in small samples.

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Appendix A

Table 1: Test for equality of the largest Lyapunov exponents from different sample sizes. Feigenbaum process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{a}_1$	$\hat{a}_1$
Block size=2	-0.0001 (-3.5657)	-0.0000 (-1.8612)
Block size=3	-0.0001 (-2.0933)	-0.0000 (-0.2914)
Block size=4	-0.0000 (-0.3795)	-0.0000 (-2.1983)
Block size=5	-0.0000 (-1.1709)	-0.0000 (-1.1001)
Block size=6	0.0000 (0.1832)	0.0000 (0.5617)

Notes:

(1) OLS estimation of the linear regression $\langle l_{\max}(T)\rangle = a_0 + a_1 T + e_T$ with t-ratio in brackets.

(2) [a] denote rejection of the null hypothesis $H_0: a_1 \leq 0$ *(deterministic process)* at the 1% level.

(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).

Table 2: Test for equality of the largest Lyapunov exponents from different sample sizes. GARCH process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{a}_1$	$\hat{a}_1$
Block size=2	0.0010 (7.4283 [a])	0.0001 (21.3217 [a])
Block size=3	0.0007 (9.6595 [a])	0.0001 (14.8049 [a])
Block size=4	0.0005 (8.4568 [a])	0.0001 (13.7545 [a])
Block size=5	0.0006 (8.3375 [a])	0.0000 (7.4032 [a])
Block size=6	0.0004 (9.8683 [a])	0.0000 (15.7122 [a])

Notes:

(1) OLS estimation of the linear regression $\langle l_{\max}(T) \rangle = a_0 + a_1 T + e_T$ with t-ratio in brackets.

(2) [a] denote rejection of the null hypothesis $H_0: a_1 \leq 0$ *(deterministic process)* at the 1% level.

(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).

Table 3: Test for equality of the largest Lyapunov exponents from different sample sizes. NLMA process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{a}_1$	$\hat{a}_1$
Block size=2	0.0011 (13.2308 [a])	0.0002 (30.3189 [a])
Block size=3	0.0004 (5.5576 [a])	0.0001 (36.2851 [a])
Block size=4	0.0002 (3.6952 [a])	0.0001 (21.8271 [a])
Block size=5	0.0003 (11.6596 [a])	0.0001 (15.0050 [a])
Block size=6	0.0004 (9.5941 [a])	0.0001 (9.6585 [a])

Notes:

(1) OLS estimation of the linear regression $\langle l_{\max}(T) \rangle = a_0 + a_1 T + e_T$ with t-ratio in brackets.

(2) [a] denote rejection of the null hypothesis $H_0: a_1 \leq 0$ *(deterministic process)* at the 1% level.

(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).

Table 4: Test for equality of the largest Lyapunov exponents from different sample sizes. ARCH process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{a}_1$	$\hat{a}_1$
Block size=2	0.0093 (11.0229 [a])	0.0001 (12.4659 [a])
Block size=3	0.0001 (13.9478 [a])	0.0001 (18.5681 [a])
Block size=4	0.0001 (9.7114 [a])	0.0001 (11.5503 [a])
Block size=5	0.0004 (8.1143 [a])	0.0000 (10.4566 [a])
Block size=6	0.0003 (8.9532 [a])	0.0001 (17.8911 [a])

Notes:

(1) OLS estimation of the linear regression $\langle l_{\max}(T) \rangle = a_0 + a_1 T + e_T$ with t-ratio in brackets.

(2) [a] denote rejection of the null hypothesis $H_0 : a_1 \leq 0$ *(deterministic process)* at the 1% level.

(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).

Table 5: Test for equality of the largest Lyapunov exponents from different sample sizes. ARMA process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{a}_1$	$\hat{a}_1$
Block size=2	0.0008 (16.0477 [a])	0.0001 (37.3805 [a])
Block size=3	0.0007 (6.8056 [a])	0.0001 (13.3807 [a])
Block size=4	0.0005 (4.6824 [a])	0.0001 (15.2237 [a])
Block size=5	0.0002 (3.6430 [a])	0.0001 (15.5015 [a])
Block size=6	0.0003 (6.4041 [a])	0.0001 (10.3826 [a])

Notes:

(1) OLS estimation of the linear regression $\langle l_{\max}(T) \rangle = a_0 + a_1 T + e_T$ with t-ratio in brackets.

(2) [a] denote rejection of the null hypothesis $H_0 : a_1 \leq 0$ *(deterministic process)* at the 1% level.

(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).

Table 6: Test for equality of the largest Lyapunov exponents from different sample sizes. Hénon process (1) (2).		
Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{a}_1$	$\hat{a}_1$
Block size=2	0.0001 (1.7765)	0.0000 (2.5822)
Block size=3	0.0000 (0.2710)	0.0000 (1.7611)
Block size=4	-0.0003 (-5.9803)	-0.0000 (-2.4872)
Block size=5	0.0000 (1.5134)	0.0000 (0.8518)
Block size=6	0.0001 (2.1635)	0.0000 (0.0867)

Notes:

(1) OLS estimation of the linear regression $\langle l_{\max}(T)\rangle = a_0 + a_1 T + e_T$ with t-ratio in brackets.

(2) [a] denote rejection of the null hypothesis $H_0: a_1 \leq 0$ *(deterministic process)* at the 1% level.

(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).

Table 7: Test for equality of the largest Lyapunov exponents from different sample sizes. Lorenz process (1) (2).

Sample size	T=380 (3)	T=2000 (3)
Coefficients of linear regression	$\hat{a}_1$	$\hat{a}_1$
Block size=2	0.0005 (7.1435[a])	0.0001 (10.4912[a])
Block size=3	0.0001 (2.1601)	0.0000 (0.9722)
Block size=4	-0.0001 (-1.0223)	0.0000 (0.3579)
Block size=5	0.0002 (1.9083)	0.0000 (2.3860)
Block size=6	0.0004 (4.1590[a])	0.0000 (2.2246)

Notes:

(1) OLS estimation of the linear regression $\langle l_{\max}(T)\rangle = a_0 + a_1 T + e_T$ with t-ratio in brackets.

(2) [a] denote rejection of the null hypothesis $H_0: a_1 \leq 0$ *(deterministic process)* at the 1% level.

(3) The critical value for the t-statistic at the 1% level for T=380 (T=2000) is 2.896 (2.821).

Appendix B

Table 8: Results of Bask and Gençay (1998)´s test. Feigenbaum process.

Sample size	T=380 (1)				T=2000 (1)			
	$\hat{l}_1$	Critical value $\hat{l}_1 - q(99\%)$	Critical value $\hat{l}_1 - q(97.5\%)$	Critical value $\hat{l}_1 - q(95\%)$	$\hat{l}_1$	Critical value $\hat{l}_1 - q(99\%)$	Critical value $\hat{l}_1 - q(97.5\%)$	Critical value $\hat{l}_1 - q(95\%)$
Block size=2	0.2599	0.1850	0.1943	0.2007	0.2714	0.1984	0.2007	0.2073
Block size=3	0.2598	0.1612	0.1762	0.1818	0.2699	0.1853	0.1946	0.1958
Block size=4	0.1629	0.0833	0.0838	0.0850	0.1692	0.0775	0.0813	0.0877
Block size=5	0.0494	-0.1023	-0.0968	-0.0883	0.0514	-0.1042	-0.1012	-0.0979
Block size=6	0.0463	-0.1198	-0.0894	-0.0795	0.0482	-0.1054	-0.0966	-0.0787

Note:

(1) $\hat{l}_1$ is an estimation of the largest Lyapunov exponent, and q(.) is the quartile for the empirical distribution formed by calculating $\tilde{l}_1 - \hat{l}_1$.
$\tilde{l}_1$ is the largest exponent from the bootstrap sample. The number of boostrap values is 100.

Table 9: Results of Bask and Gençay (1998)´s test. Garch process.								
Sample size	T=380 (1)				T=2000 (1)			
Garch	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$
Block size=2	1.3036	1.2115	1.2153	1.2248	1.7089	1.6447	1.6618	1.6676
Block size=3	0.8173	0.7028	0.7554	0.7575	1.0555	1.0133	1.0250	1.0275
Block size=4	0.5901	0.5135	0.5208	0.5264	0.7447	0.6998	0.7092	0.7116
Block size=5	0.4318	0.3362	0.3558	0.3874	0.5323	0.4982	0.5001	0.5037
Block size=6	0.3090	0.2280	0.2471	0.2569	0.4081	0.3657	0.3779	0.3845

Note:

(1) $\hat{\lambda}_1$ is an estimation of the largest Lyapunov exponent, and q(.) is the quartile for the empirical distribution formed by calculating $\tilde{\lambda}_1 - \hat{\lambda}_1$. $\tilde{\lambda}_1$ is the largest exponent from the bootstrap sample. The number of boostrap values is 100.

Table 10: Results of Bask and Gençay (1998)´s test. Nlma process.								
Sample size	T=380 (1)				T=2000 (1)			
	$\hat{l}_1$	Critical value $\hat{l}_1 - q(99\%)$	Critical value $\hat{l}_1 - q(97.5\%)$	Critical value $\hat{l}_1 - q(95\%)$	$\hat{l}_1$	Critical value $\hat{l}_1 - q(99\%)$	Critical value $\hat{l}_1 - q(97.5\%)$	Critical value $\hat{l}_1 - q(95\%)$
Block size=2	1.2835	1.1745	1.2085	1.2152	1.6693	1.5954	1.6148	1.6191
Block size=3	0.7968	0.6887	0.7299	0.7484	1.0579	1.0070	1.0206	1.0245
Block size=4	0.5234	0.4419	0.4465	0.4589	0.7364	0.6962	0.7041	0.7091
Block size=5	0.4048	0.2908	0.3442	0.3567	0.5364	0.4868	0.4947	0.5001
Block size=6	0.3188	0.2474	0.2568	0.2689	0.4070	0.3682	0.3780	0.3832

Note:

(1) $\hat{l}_1$ is an estimation of the largest Lyapunov exponent, and q(.) is the quartile for the empirical distribution formed by calculating $\tilde{l}_1 - \hat{l}_1$.

$\tilde{l}_1$ is the largest exponent from the bootstrap sample. The number of boostrap values is 100.

Table 11: Results of Bask and Gençay (1998)´s test. Arch process.								
Sample size	T=380 (1)				T=2000 (1)			
	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$	$\hat{\lambda}_1$	Critical value $\hat{\lambda}_1 - q(99\%)$	Critical value $\hat{\lambda}_1 - q(97.5\%)$	Critical value $\hat{\lambda}_1 - q(95\%)$
Block size=2	1.3125	1.1971	1.2039	1.2198	1.6506	1.5757	1.5943	1.6003
Block size=3	0.8143	0.7192	0.7375	0.7525	1.0706	1.0228	1.0324	1.0350
Block size=4	0.5571	0.4444	0.4799	0.4974	0.7440	0.6878	0.6978	0.7142
Block size=5	0.4042	0.3196	0.3318	0.3463	0.5411	0.5029	0.5050	0.5089
Block size=6	0.2954	0.2172	0.2337	0.2426	0.4135	0.3647	0.3724	0.3817

Note:

(1) $\hat{\lambda}_1$ is an estimation of the largest Lyapunov exponent, and q(.) is the quartile for the empirical distribution formed by calculating $\tilde{\lambda}_1 - \hat{\lambda}_1$.
$\tilde{\lambda}_1$ is the largest exponent from the bootstrap sample. The number of boostrap values is 100.

Table 12: Results of Bask and Gençay (1998)´s test. Arma process								
Sample size	T=380 (1)				T=2000 (1)			
	$\hat{l}_1$	Critical value $\hat{l}_1 - q(99\%)$	Critical value $\hat{l}_1 - q(97.5\%)$	Critical value $\hat{l}_1 - q(95\%)$	$\hat{l}_1$	Critical value $\hat{l}_1 - q(99\%)$	Critical value $\hat{l}_1 - q(97.5\%)$	Critical value $\hat{l}_1 - q(95\%)$
Block size=2	1.1631	1.0760	1.0861	1.0941	1.5186	1.4851	1.4868	1.4911
Block size=3	0.7374	0.6380	0.6452	0.6625	0.9749	0.9249	0.9408	0.9449
Block size=4	0.4927	0.4062	0.4227	0.4262	0.7030	0.6599	0.6673	0.6740
Block size=5	0.3746	0.3023	0.3063	0.3116	0.5351	0.4919	0.4951	0.5004
Block size=6	0.2958	0.1798	0.2196	0.2287	0.4137	0.3810	0.3893	0.3911

Note:

(1) $\hat{l}_1$ is an estimation of the largest Lyapunov exponent, and q(.) is the quartile for the empirical distribution formed by calculating $\tilde{l}_1 - \hat{l}_1$.
$\tilde{l}_1$ is the largest exponent from the bootstrap sample. The number of boostrap values is 100.

Table 13: Results of Bask and Gençay (1998)´s test. Hénon process								
Sample size	T=380 (1)				T=2000 (1)			
	$\hat{l}_1$	Critical value $\hat{l}_1 - q(99\%)$	Critical value $\hat{l}_1 - q(97.5\%)$	Critical value $\hat{l}_1 - q(95\%)$	$\hat{l}_1$	Critical value $\hat{l}_1 - q(99\%)$	Critical value $\hat{l}_1 - q(97.5\%)$	Critical value $\hat{l}_1 - q(95\%)$
Block size=2	0.4154	0.3005	0.3563	0.3653	0.4407	0.4083	0.4152	0.4158
Block size=3	0.3686	0.2887	0.2958	0.3041	0.3953	0.3652	0.3713	0.3727
Block size=4	0.3637	0.2857	0.2980	0.3038	0.3734	0.3404	0.3501	0.3514
Block size=5	0.3836	0.2971	0.3096	0.3221	0.3853	0.3443	0.3581	0.3603
Block size=6	0.3825	0.3033	0.3135	0.3270	0.3831	0.3557	0.3598	0.3606

Note:

(1) $\hat{l}_1$ is an estimation of the largest Lyapunov exponent, and q(.) is the quartile for the empirical distribution formed by calculating $\tilde{l}_1 - \hat{l}_1$.

$\tilde{l}_1$ is the largest exponent from the bootstrap sample. The number of boostrap values is 100.

Table 14: Results of Bask and Gençay (1998)´s test. Lorenz process.								
Sample size	T=380 (1)				T=2000 (1)			
	$\hat{l}_1$	Critical value $\hat{l}_1 - q(99\%)$	Critical value $\hat{l}_1 - q(97.5\%)$	Critical value $\hat{l}_1 - q(95\%)$	$\hat{l}_1$	Critical value $\hat{l}_1 - q(99\%)$	Critical value $\hat{l}_1 - q(97.5\%)$	Critical value $\hat{l}_1 - q(95\%)$
Block size=2	0.9234	0.7842	0.8067	0.8197	1.2207	1.1520	1.1764	1.1800
Block size=3	0.6417	0.4788	0.5200	0.5408	0.7427	0.6784	0.6952	0.6999
Block size=4	0.5010	0.3051	0.3914	0.4071	0.5680	0.4862	0.4953	0.5020
Block size=5	0.4293	0.2590	0.2908	0.3409	0.4690	0.3900	0.4052	0.4105
Block size=6	0.3407	0.1932	0.2207	0.2270	0.3869	0.2694	0.3249	0.3277

Note:

(1) $\hat{l}_1$ is an estimation of the largest Lyapunov exponent, and q(.) is the quartile for the empirical distribution formed by calculating $\tilde{\lambda}_1 - \hat{\lambda}_1$.
$\tilde{l}_1$ is the largest exponent from the bootstrap sample. The number of boostrap values is 100.

Appendix C

Figure 1:

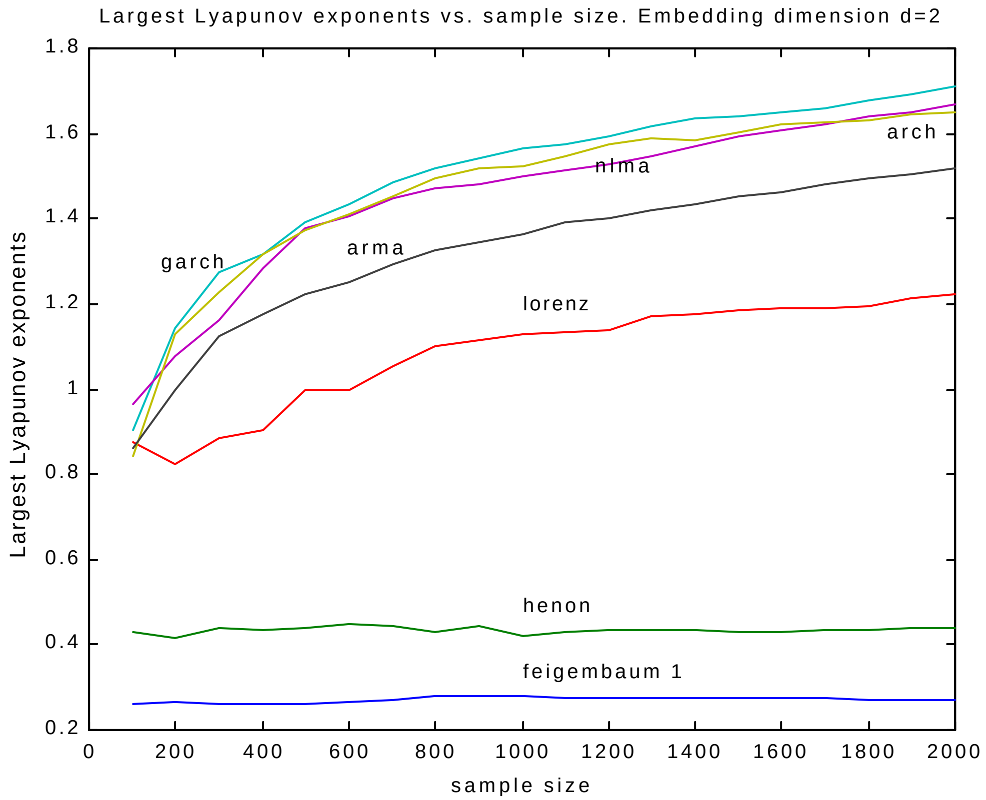


Figure 2:

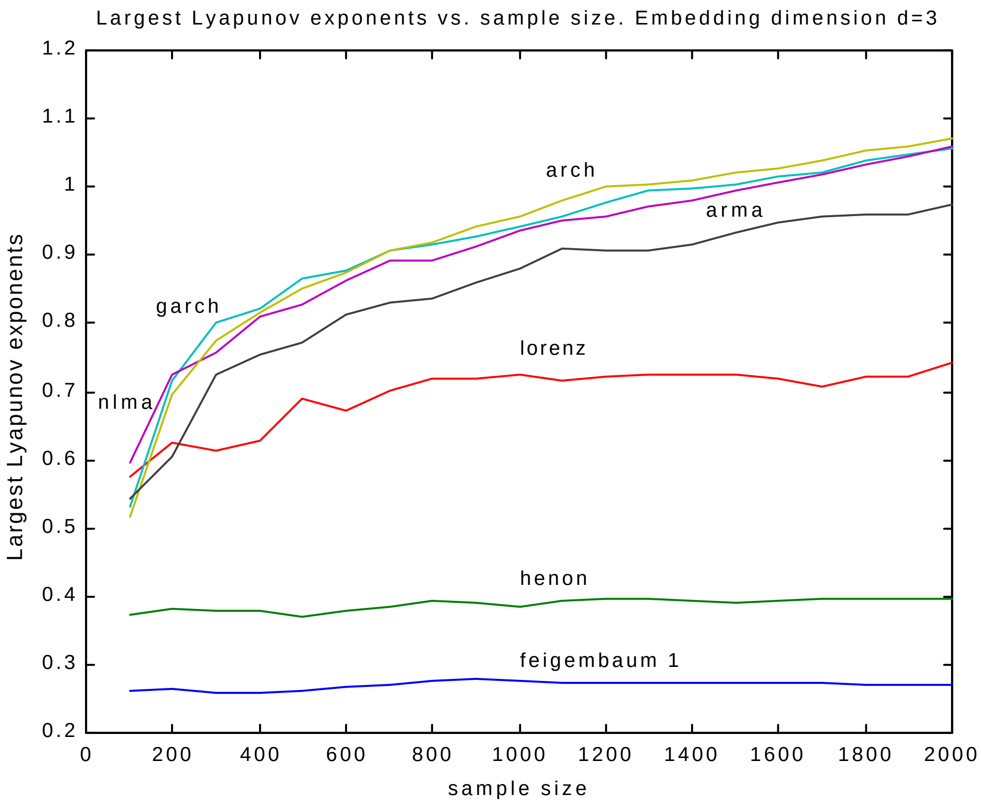
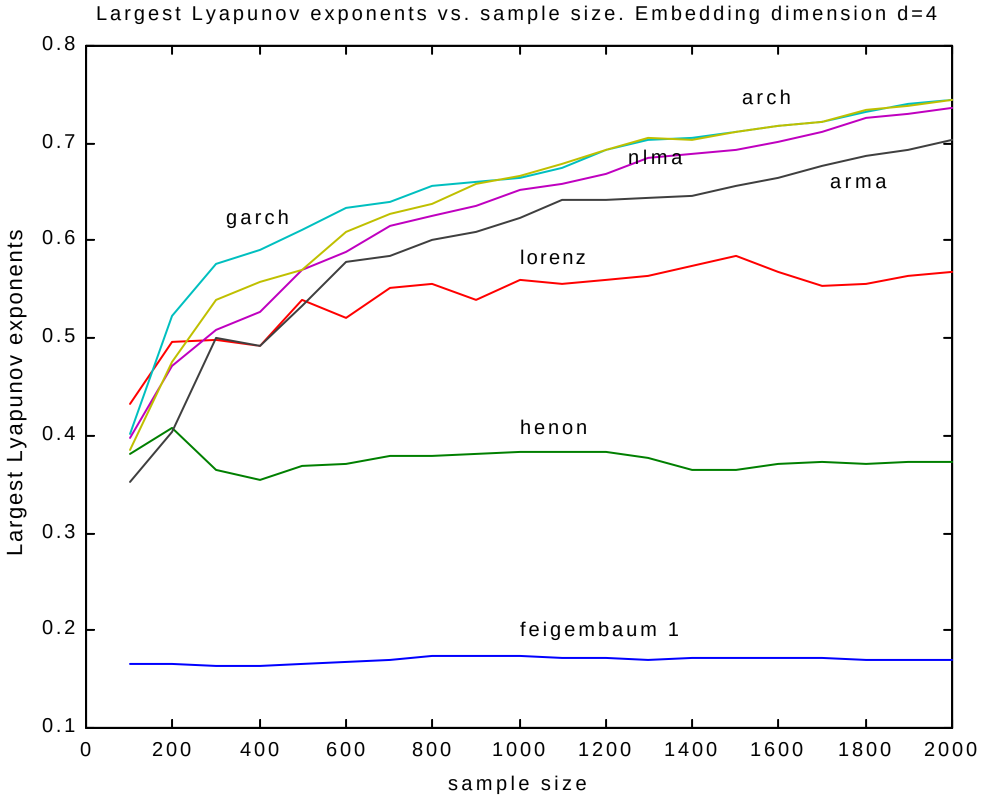

Largest Lyapunov exponents vs. sample size. Embedding dimension d=3
Largest Lyapunov exponents
sample size
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
200
400
600
800
1000
1200
1400
1600
1800
2000
arch
arma
garch
nlma
lorenz
henon
feigembaum 1

Figure 3:



Largest Lyapunov exponents vs. sample size. Embedding dimension d=4
Largest Lyapunov exponents
sample size
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
200
400
600
800
1000
1200
1400
1600
1800
2000
arch
nlma
arma
garch
lorenz
henon
feigembaum 1

Figure 4:

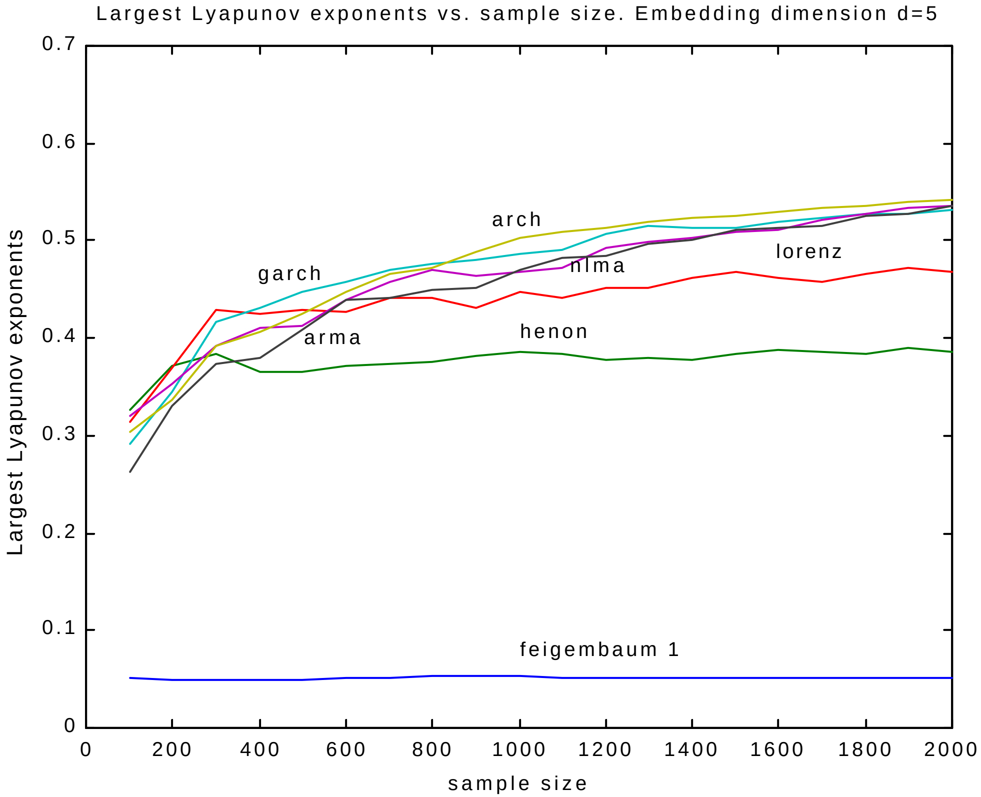
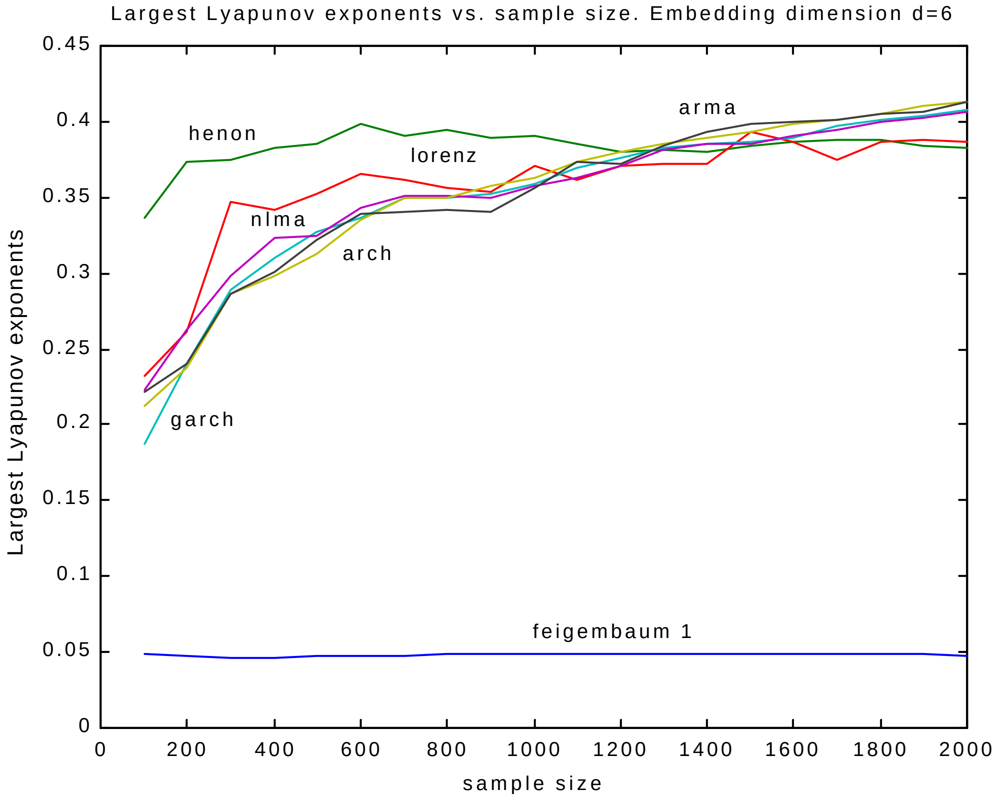

Largest Lyapunov exponents vs. sample size. Embedding dimension d=5
Largest Lyapunov exponents
sample size
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
200
400
600
800
1000
1200
1400
1600
1800
2000
arch
garch
nlma
lorenz
arma
henon
feigembaum 1

Figure 5:


Largest Lyapunov exponents vs. sample size. Embedding dimension d=6
Largest Lyapunov exponents
sample size
henon
lorenz
arma
nlma
arch
garch
feigembaum 1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0
200
400
600
800
1000
1200
1400
1600
1800
2000