Linear model based simulation of the evolution of the labor cost in the case of job training

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Abstract: The main purpose of this paper is to simulate the evolution of the labor cost in the case of job training, in the framework of a model presented in [1]. The aim is to give a numerical support for a strategy of periodic training, to maintain the "in – house" workers productivity within some limits.

Keywords: Linear model simulation, labor cost, job training.
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1. The mathematical model.

The evolution of the labor cost in the case of job training is given by the equation:
In the above equation :

$$(1)\quad C(t) = \frac{W_1}{X}\left\{Q(t) - sK + \left[\acute{a}x_0 - (\acute{a} - \hat{a}u(t))x(t)\right]L_2(t)\right\} + cu(t)L_2(t)$$

W_1 - the wage per unit of "outside" labor.

X - the productivity of contracted "outside" labor.

Q(t) - the production function given by:

$$\text{(2)}\quad Q(t) = sK + XL_1(t) + (1 - \text{â}u(t))x(t)L_2(t)$$

s - the productivity of capital (assumed to be constant).

K - the capital (assumed to be fixed).

$L_1(t)$ -the number of “outside” labor available at a competitive market.

_ - the opportunity cost of training (assumed to be constant $0 \leq \beta \leq 1$).

u(t) - the control variable which express the intensity of “in – house” labor training; $0 \leq u(t) \leq 1$.

x(t) - the productivity of “in – house” labor.

$L_2(t)$ -the number of “in-house” labor.

α - the “loyalty” coefficient (assumed to be constant $0 \leq \alpha \leq 1$).

$x_0 = x(0)$ –the initial “in-house” labor productivity.

c - is the training cost per worker per unit of time (assumed to be constant).

If u=1 then “in – house” labor is training 100% of the time and if u=0 the “in-house” labor is not training.

The evolution of the productivity x(t) is governed by the state equation:

$$\text{(3)}\quad \frac{dx}{dt} = u(t)\left[1 - \frac{x(t)}{Y} + \text{ä}x(t)\right] - \text{ä}x(t)$$

In the equation (3): Y is a constant greater than or equal to the outside–labor productivity X; $x_0 \leq X \leq Y$; δ - represents human capital depreciation in the absence of training ($\delta \geq 0$, δ -assumed to be constant).

The “in – house” wage adjustment equation is :

$$\text{(4)}\quad W_2(t) = x_0 \text{á} \frac{W_1}{X} + (1 - \text{á})x(t)\frac{W_1}{X}$$

Equations (1)-(4) define the mathematical model of the labor cost in the case of job training.

One problem is to choose the control variable u(t) which minimizes the labor cost of production Q.

The Hamiltonian of the above problem is given by [1]:

$$\text{(5)}\quad H = \frac{W_1}{X}\left[Q - sK - \text{á}(x - x_0)L_2\right] - \text{ëä}x + \left[\frac{W_1}{X}\text{â}xL_2 + cL_2 + \text{ë}(1 - \frac{1}{Y}x + \text{ä}x)\right]u =$$

$$= H_0(x,\text{ë}) + uH_1(x,\text{ë})$$

H_0 – represents the part over which the firm has no control, and H_1u - represents the part which can be influenced by the control function u.

The first order conditions for optimality are :

$$(6)\qquad \begin{cases} \dfrac{dx}{dt} = u(1-\dfrac{x}{Y}+\text{ä}x)-\text{ä}x \\ \dfrac{d\text{ë}}{dt} = \left[\left(\dfrac{1}{Y}-\text{ä}\right)u+\text{ä}\right]\text{ë}+\dfrac{W_1}{X}L_2(\text{ä}-\text{â}u) \end{cases}$$

Since the Hamiltonian (5) is linear in the control, the application of the maximum principle leads to a “bang – bang” solution :

$$(7)\ u(t)=\begin{cases} 1 \text{ if } \quad H_1<0 \\ 0 \text{ if } \quad H_1>0 \\ \text{any value in } (0,1) \text{ if } H_1=0 \end{cases}$$

2. Numerical simulation.

The values of the parameters for simulation are: W_1=10; X=0.8; β=0.1; c=0.3; L_2=500; δ=0.05; Y=1; Q=1000; α=0.8;K=100;s=1. The numerical cod used for simulation is Mathematica 4.0.

The curve defined by H_1=0 in the (x,λ) plane for x>0 is plotted in Fig.1.

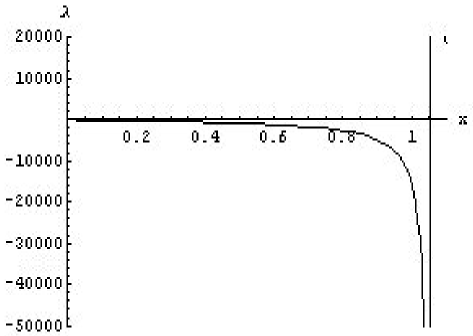


Fig.1. The curve defined by H_1=0

Case 1. The productivity x(0) satisfies $X<x(0)$ and $u(0)=0$.

The solution of (6) corresponding to the initial conditions $x(0)=0.9$ and $\lambda(0)=-2000$ is plotted in Fig.2.

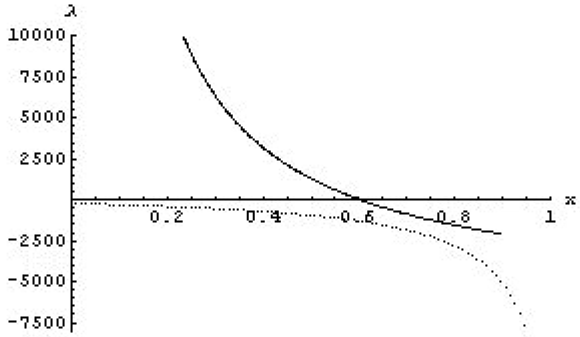


Fig.2. Solution of (6) corresponding to the initial condition x(0)=0.9 and _(0)=-2000.

In this computation $u(0)=0$ because $H_1(x(0),\lambda(0))>0$. Computation shows that: for $t>0$ $H_1(x(t),\lambda(t))>0$ and therefore $u(t)=0$ for $t>0$. The curve $(x(t),\lambda(t))$ does not intersect the curve defined by $H_1=0$. The "in – house" productivity x(t) decreases and tends to 0 for t_∞.

The computed evolution of the value of the labor cost C is plotted in Fig.3.

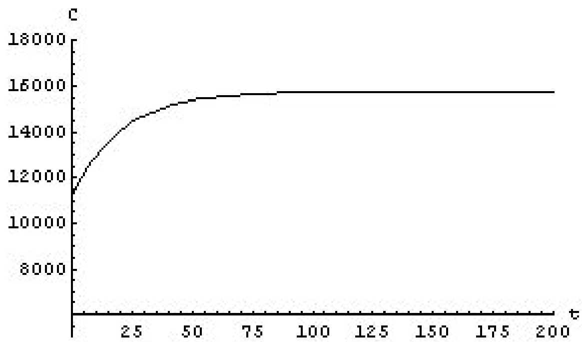


Fig.3. The evolution of the labor cost.

Computation shows that the labor cost increases from $C(0)=11250$ to the steady cost $C(\infty)=15750$ given by

$$(8)\quad C(\infty)=\frac{W_1}{X}(Q-sK)+\frac{W_1}{X}x_0 \acute{a} L_2$$

The computed evolution of the "outside" labor cost C_e is plotted in Fig.4.

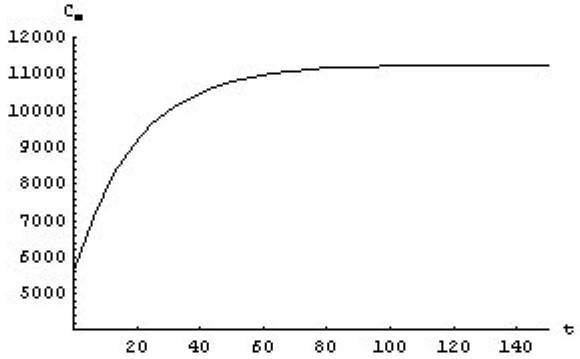


Fig.4. The evolution of the "outside" labor cost.

Computation shows that the "outside" labor cost C_e increases from $C_e(0)=5625$ to $C_e(\infty)=11250$.

The computed evolution of the "in-house" labor cost C_i is plotted in Fig.5.

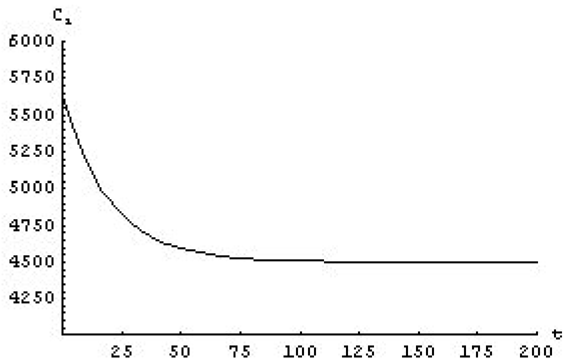


Fig.5. The evolution of the "in-house" labor cost.

Computation shows that the "in-house" labor cost C_i decreases from $C_i(0)=5625$ to $C_i(\infty)=\frac{W_1}{X}x_0\acute{a}L_2=4500$.

The computed evolution of the number of the "outside" labors is plotted in Fig.6.

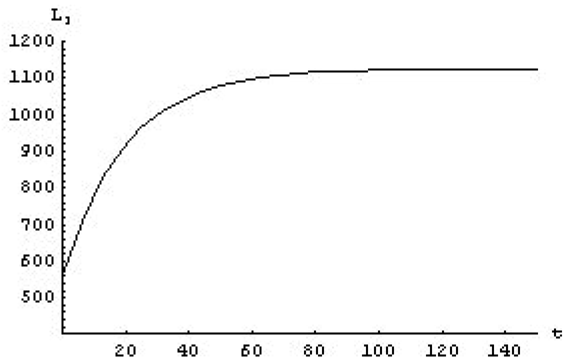


Fig.6. The evolution of the number of "outside" labors.

Computation shows that the number of "outside" labors increases from $L_1(0)=562$ to $L_1(\infty)=1125$.

The computed evolution of production Q in this case is plotted in Fig.7.

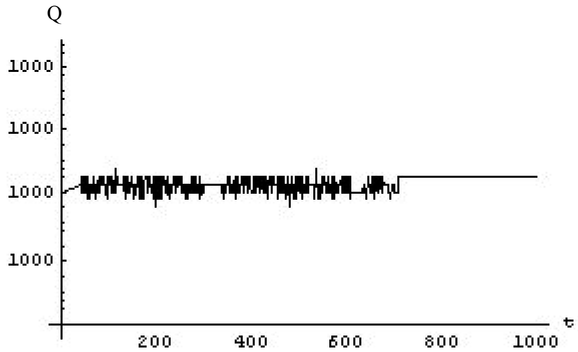


Fig.7. The evolution of the production Q.

Computation shows that the production Q(t) is constant and equal to 1000 .

The computed production Q_i of the "in-house" labor is plotted in Fig.8.

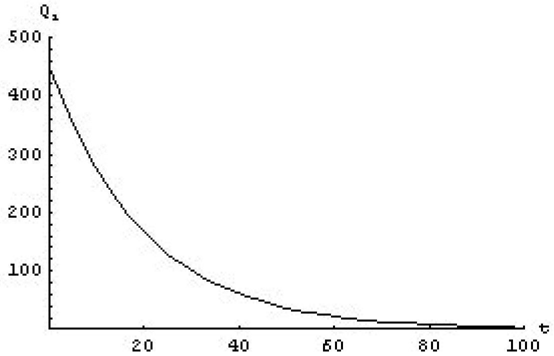


Fig.8. The evolution of the production of the "in – house" labor.

Computation shows that Q_i decreases from $Q_i(0)=450$ to $Q_i(\infty)=0$.

The computed production Q_e of the "outside" labor is plotted in Fig.9.

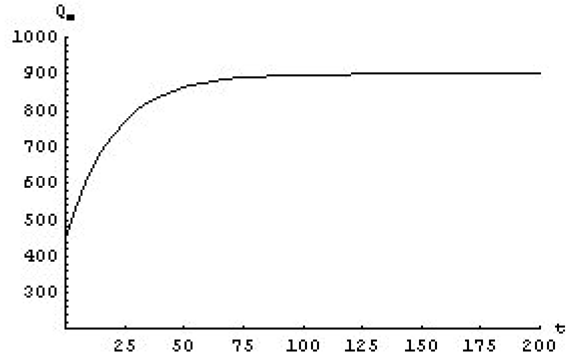


Fig.9. The evolution of the production of the "outside" labor.

Computation shows that Q_e increases from $Q_e(0)=450$ to $Q_e(\infty)=900$.

Case 2. The productivity x(0) satisfies X<x(0) and u(0)=1.

For the initial conditions x(0)=0 and _(0)=-7500 the solution of (6) is plotted in Fig. 10.

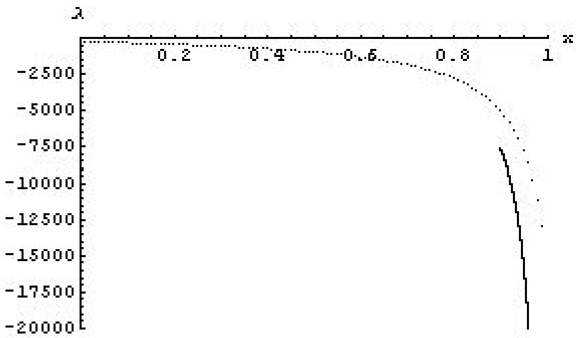


Fig.10. Solution of (6) corresponding to the initial condition x(0)=0.9 and _(0)=-7500.

In this computation u(0)=1 because $H_1(x(0),\lambda(0))<0$. Computation shows that: for t>0 $H_1(x(t),\lambda(t))<0$ and therefore u(t)=1 for t>0. The curve $(x(t),\lambda(t))$ does not intersect the curve defined by $H_1=0$. The "in – house" productivity x(t) increases and tends to 1 for t_∞.

The computed evolution of the value of the labor cost C is plotted in Fig.11.

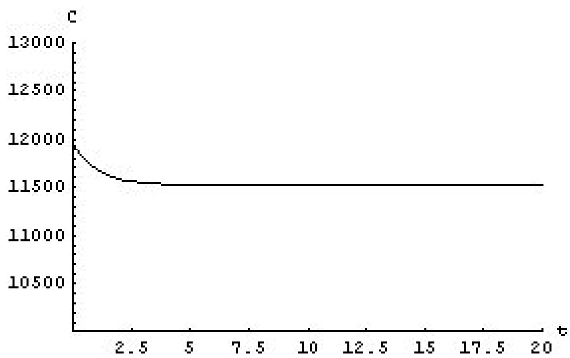


Fig.11. The evolution of the labor cost.

Computation shows that the labor cost decreases from C(0)=11962 given by

$$(9)\quad C(0) = \frac{W_1}{X}(Q - sK) + \frac{W_1}{X}\hat{a}x_0L_2 + cL_2$$

to C(∞)=11525 given by

$$(10)\quad C(\infty) = \frac{W_1}{X}(Q - sK) + \frac{W_1}{X}(\acute{a}\acute{a}_0 - \acute{a} + \hat{a})L_2 + cL_2$$

The term $\frac{W_1}{X}(Q - sK)$=11250 represents the cost of the "outside" labor, the term $\frac{W_1}{X}\hat{a}x_0L_2$=562 represents the opportunity cost of training and the term cL_2=150 represents the direct cost of training.

The computed evolution of the "outside" labor cost C_e is plotted in Fig.12.

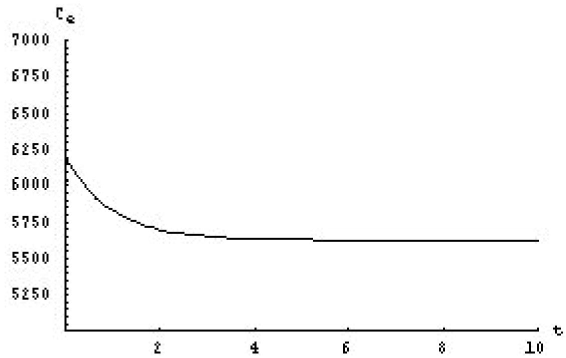


Fig.12. The evolution of the "outside" labor cost.

Computation shows that the "outside" labor cost C_e decreases from $C_e(0)$=6180 to $C_e(\infty)$=5620.

The computed evolution of the "in-house" labor cost C_i is plotted in Fig.13.

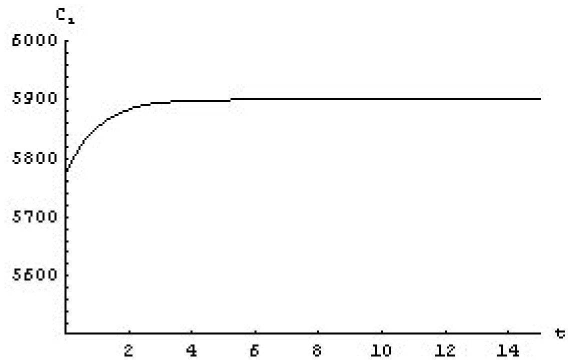


Fig.13. The evolution of the "in - house" labor cost.

Computation shows that the "in – house" labor cost C_i increases from $C_i(0)$ =5775 to $C_i(\infty)$=5900.

The computed evolution of the number of the "outside" labors is plotted in Fig.14.

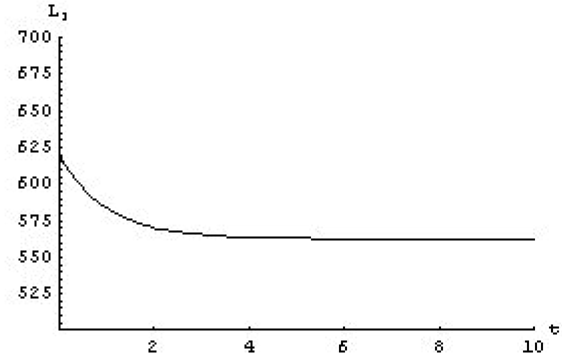


Fig.14. The evolution of the number of the "outside" labors.

Computation shows that the number of the "outside" labors decreases from $L_1(0)$=618 to $L_1(\infty)$=562.

The computed evolution of the production is plotted in Fig.15.

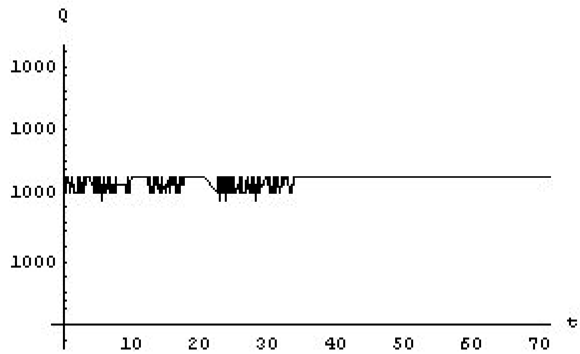


Fig.15. The evolution of the production

Computation shows that the production Q is constant and equal to 1000.

The computed evolution of the production of the “in – house” labor is plotted in Fig.16.

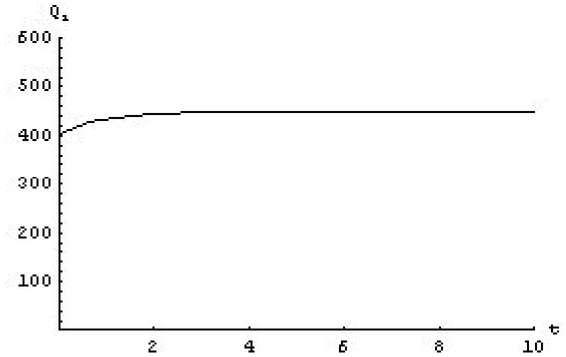


Fig.16. The evolution of the production of the “in – house” labor.

Computation shows that Q_i increases from $Q_i(0)=405$ to $Q_i(\infty)=450$.

The computed evolution of the production of the “outside “ labor is plotted in Fig.17.

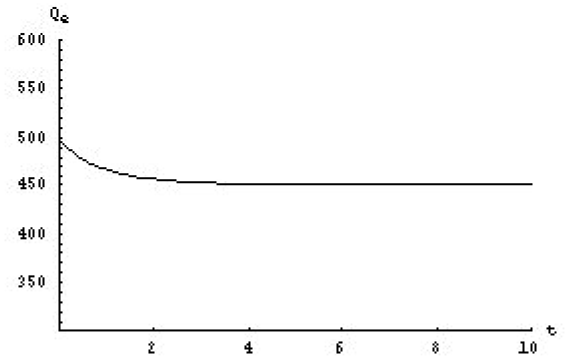


Fig.17. The evolution of the production of the “outside” labor.

Computation shows that Q_e decreases from $Q_e(0)=495$ to $Q_e(\infty)=450$.

Case3. The productivity x(0) satisfies x(0)<X and u(0)=1.

For the initial conditions x(0)=0.2 and λ(0)=-1500 the solution of (6) is plotted in Fig.18.

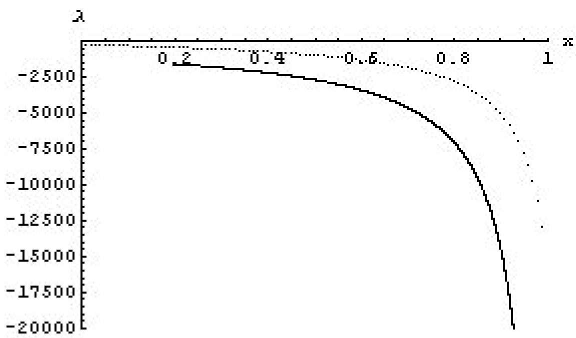


Fig.18. The solution of (6) corresponding to the initial condition x(0)=0.2 and _(0)=-1500

In this computation u(0)=1 because $H_1(x(0),\lambda(0))<0$. Computation shows that for t>0 $H_1(x(t),\lambda(t))<0$ and therefore u(t)=1 for t>0. The curve $(x(t),\lambda(t))$ does not intersect the curve defined by $H_1=0$. The "in – house" productivity x(t) increases and tends to 1 for t_∞.

The computed evolution of the value of the labor cost is plotted in Fig.19.

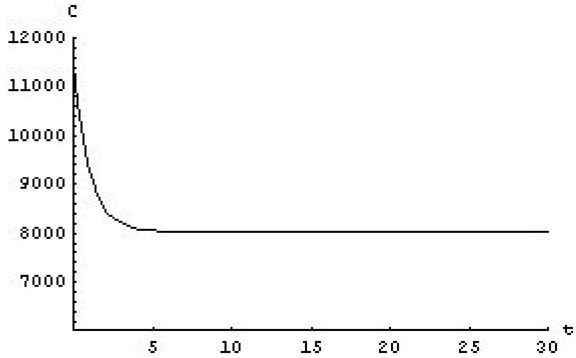


Fig.19. The evolution of the value of the labor cost.

Computation shows that the labor cost decreases from C(0)=11525, given by:

$$\text{(11)}\quad C(0) = \frac{W_1}{X}(Q - sK) + \frac{W_1}{X}\text{â}x_0L_2 + cL_2,$$

to C (∞)=8025, given by

$$\text{(12)}\quad C(\infty) = \frac{W_1}{X}(Q - sK) + \frac{W_1}{X}(\text{áá}_0 - \text{á} + \text{â})L_2 + cL_2$$

The term $\frac{W_1}{X}(Q - sK)$=11250 represents the cost of production of the "outside" labor, the term $\frac{W_1}{X}\text{â}x_0L_2$=125 represents the opportunity cost of training and the term cL_2=150 represents the direct cost of training.

The computed evolution of the "outside" labor cost C_e is plotted in Fig.20.

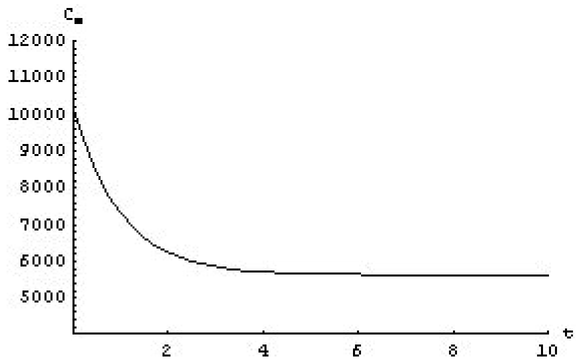


Fig.20. The evolution of the "outside" labor cost.

Computation shows that the "outside" labor cost C_e decreases from $C_e(0)=10120$ to $C(\infty)=5620$.

The computed evolution of the "in-house" labor cost C_i is plotted in Fig.21.

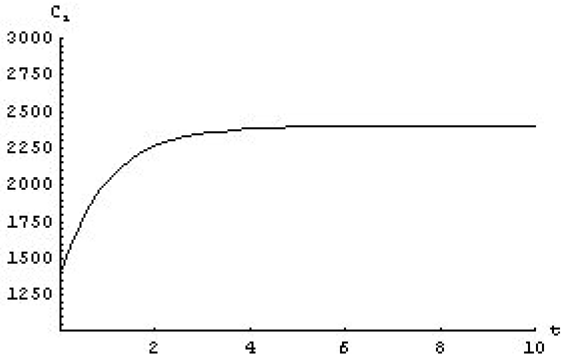


Fig.21. The evolution of the "in - house" labor cost.

Computation shows that the "in – house" labor cost C_i increases from $C_i(0)=1400$ to $C_i(\infty)=2400$.

The computed evolution of the number of the "outside" labors is plotted in Fig.22.

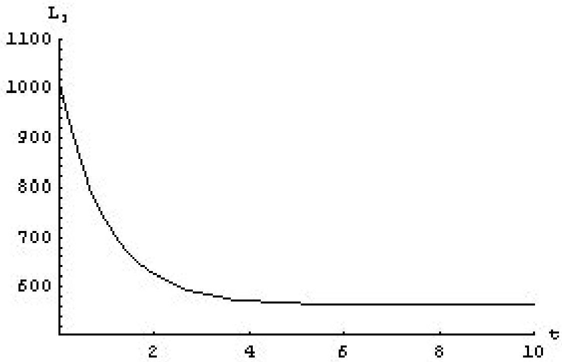


Fig.22. The evolution of the number of "outside" labors.

Computation shows that the number of the "outside" labors decreases from $L_1(0)=1012$ to $L_1(\infty)=562$.

The computed evolution of production is plotted in Fig.23.

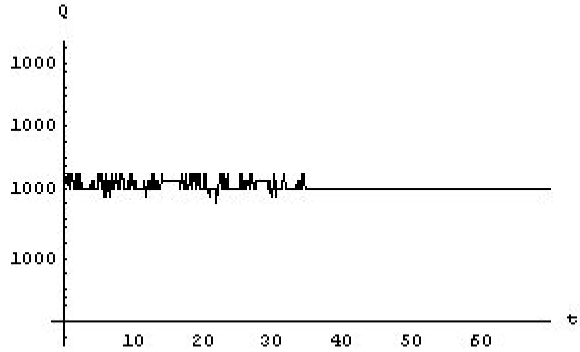


Fig.23. The evolution of the production.

Computation shows that the production Q is constant and equal to 1000.

The computed evolution of the production of the "in – house" labor is plotted in Fig.24.

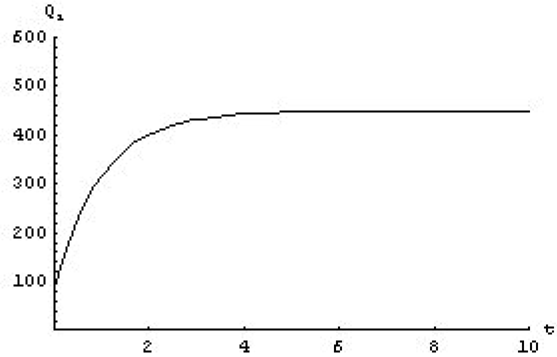


Fig.24. The evolution of the production of the "in – house" labor.

Computation shows that Q_i increases from $Q_i(0)=90$ to $Q_i(\infty)=450$.

The computed evolution of the production of the "outside " labor is plotted in Fig.25.

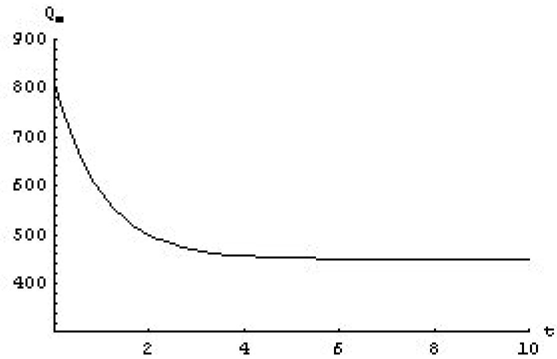


Fig.25. The evolution of the production of the "outside" labor.

Computation shows that Q_e decreases from $Q_e(0)=810$ to $Q_e(\infty)=450$.

Case 4. The productivity x(0) satisfies x(0)<X and u(0)=0.

For the initial conditions x(0)=0.2 and _(0)=5000 the solution of (6) is plotted in Fig.26.

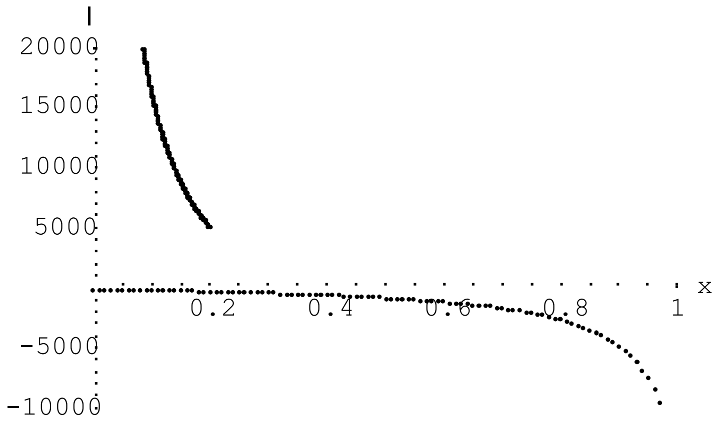


Fig.26. The solution of (6) corresponding to the initial condition x(0)=0.2 and _(0)=5000. In this computation u(0)=0, because $H_1(x(0),\lambda(0))>0$. Computation shows that: for t>0 $H_1(x(t),\lambda(t))>0$ and therefore u(t)=0 for t>0. The curve $(x(t),\lambda(t))$ does not intersect the curve defined by $H_1=0$. The "in – house" productivity x(t) decreases and tends to 0 for t_∞.

The computed evolution of the value of the labor cost C is plotted in Fig.27.

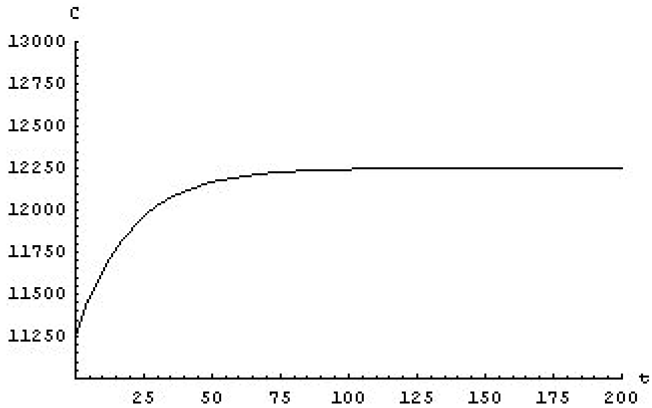


Fig.27. The evolution of the value of the labor cost.

Computation shows that the labor cost increases from C(0)=11250 to the steady cost C(∞)=12250, given by

$$(13)\quad C(\infty)=\frac{W_1}{X}(Q-sK)+\frac{W_1}{X}x_0 áL_2$$

The computed evolution of the “outside” labor cost C_e is plotted in Fig.28.

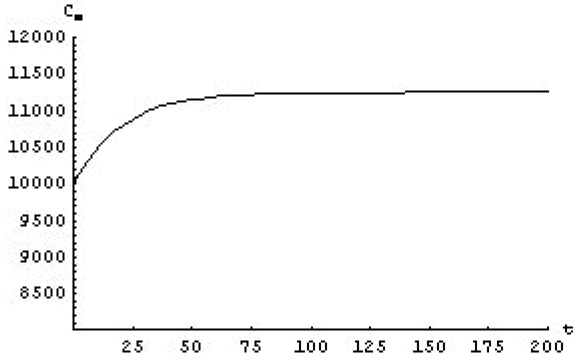


Fig.28. The evolution of the “outside” labor cost.

Computation shows that the “outside” labor cost C_e increases from $C_e(0)=10120$ to $C_e(\infty)=11250$.

The computed evolution of the “in-house” labor cost C_i is plotted in Fig.29.

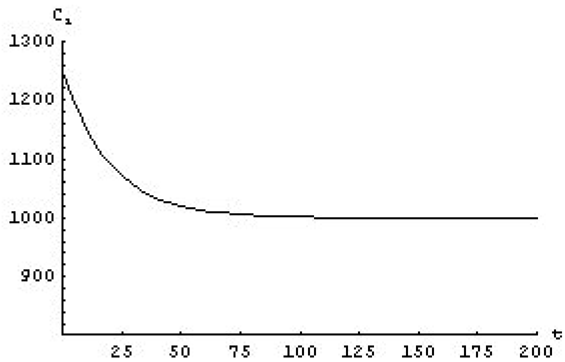


Fig.29. The evolution of the “in-house” labor cost.

Computation shows that the “in-house” labor cost C_i decreases from $C_i(0)=1250$ to $C_i(\infty)=\frac{W_1}{X}x_0 \acute{a} L_2=1000$.

The computed evolution of the number of the “outside” labors is plotted in Fig.30.

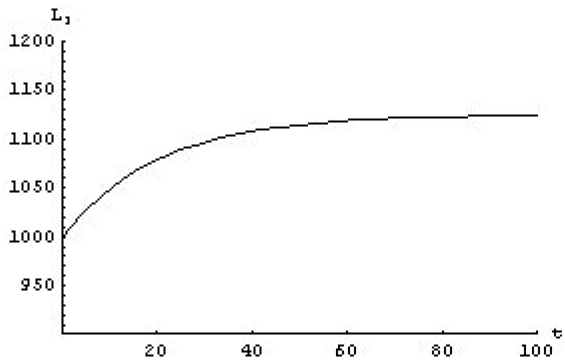


Fig.30. The evolution of the numbers of the “outside” labor.

Computation shows that the number of "outside" labors increases from $L_1(0)=1000$ to $L_1(\infty)=1125$.

The computed evolution of production Q in this case is plotted in Fig.31.

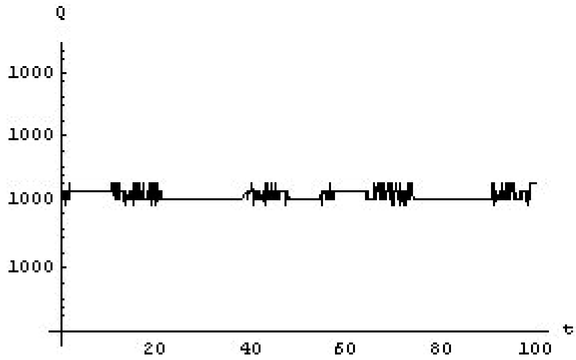


Fig.31. The evolution of the production

Computation shows that the production Q(t) is constant and equal to 1000 .

The computed production Q_i of the "in-house" labor is plotted in Fig.32.

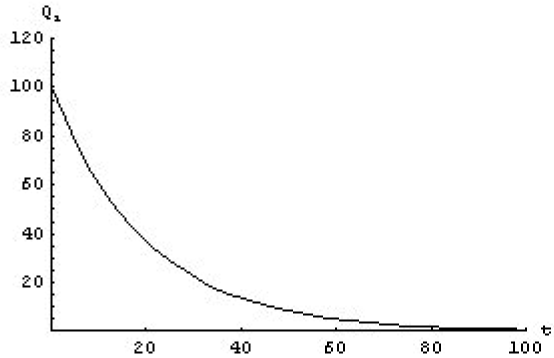


Fig.32. The evolution of the production of the "in – house" labor.

Computation shows that Q_i decreases from $Q_i(0)=100$ to $Q_i(\infty)=0$.

The computed production Q_e of the "outside" labor is plotted in Fig.33.

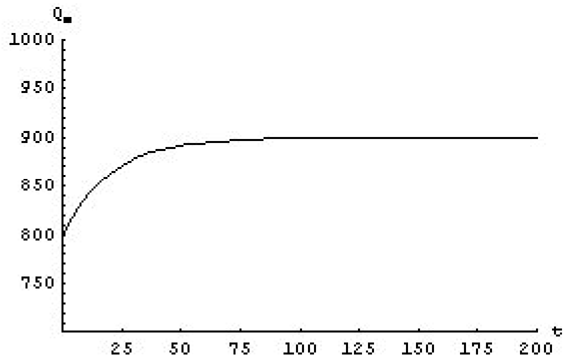


Fig.33. The evolution of the production of the "outside" labor.

Computation shows that Q_e increases from $Q_e(0)=800$ to $Q_e(\infty)=900$.

Conclusions:

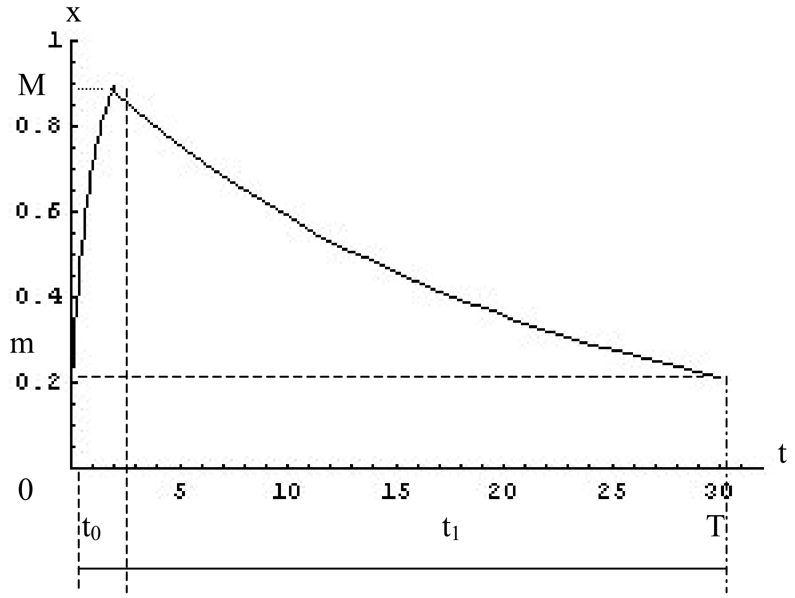
1) $c < \frac{W_1}{X}(á - â)$ and training determines the increase of the “in - house” productivity and the decrease of the labor cost at the minimal level.

2) $x_0 > X$ and training determines the increase of the labor cost.

3) An option is to use periodic training to maintain the productivity of the “in-house” workers within some limits. The firm alternates between periods of training (u=1) and no training (u=0). In the training period, productivity is raised up to maximum level denoted by M and in the following period , productivity is allowed to decay to a minimum level denoted by m.

The length of the training period, denoted by t_0 is given by: $t_0 = \ln(\frac{1-m}{1-M})$ and the length of the non-training period, denoted by t_1, is given by: $t_1 = \frac{1}{ä}\ln(\frac{M}{m})$.

The time path of productivity for m=0.2 and M=0.9 is shown in figure 8:



The costs during the period of training, denoted by C_{00}, is obtained by integrating the cost function over the interval $[0, t_0]$ and the costs during the period of non training is obtained by integrating the cost function over the interval $[0,t_1]$, respectively.

The cost per unit of time over the period [0,T], (where $T=t_0+t_1$) , denoted by C_2, is :

$$C_2 = C_0 + \frac{1}{T}\left(cL_2 \ln\left[\frac{1-m}{1-M}\right]\right) + \hat{a}L_2 \frac{W_1}{XT}\left\{ \ln\left[\frac{1-m}{1-M}\right] - (M-m)\right\} +$$

$$+ \acute{a}L_2 \frac{W_1}{XT}\left\{ \left(1-\ddot{a}^{-1}\right)(M-m) - \ln\left[\frac{1-m}{1-M}\right]\right\}$$

The first term on the right hand side of the cost C_2 is the labor costs without training, the second term is the direct cost of training, the third term is the indirect cost of training and the last term is the cost saving generated by worker loyalty.

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