Controllability of the income and consumption growth in a Keynesian macro economical model

M.C.Voicu[1*], I.Mihai[1] A.M.Balint[2] , _t.Balint[3]

1. Department of Economics, University of the West Timi_oara, Str. Pestalozzi No.16, 1900, Timi_oara, România

2. Department of Physics, University of the West Timi_oara, Blv. V. Pârvan No.4, 1900, Timi_oara, România

3. Department of Mathematics, University of the West Timi_oara, Blv. V. Pârvan No.4, 1900, Timi_oara, România

* Corresponding author: M.C.Voicu, Department of Economics, University of the West Timi_oara, Str. Pestalozzi No.16, 1900, Timi_oara, România, Tel/Fax: +40 56 190 698, E-mail: gabi@fse1.uvt.ro

Abstract. The main purpose of this paper is to prove that in the framework of a Keynesian macroeconomic model the income and the consumption can be controlled modifying the coefficients of the consumption function.

Keywords: Controllability, income, consumption

A.M.S. (M.O.S) Subject classification: 34.H.05

1. A Keynesian macroeconomic model

Let be a macroeconomic model in which the income Y rises in response to excess aggregate demand (D - Y). We assume that the aggregate demand D is equal to C + I + G where C is the consumption, I is the investment and G is the government expenditure (D = C + I + G). We assume also that investment I and government expenditure G are positive constants and consumption C is a positive increasing function of Y, C = C(Y). The diversification of consumption in function of the income (fixed consumption independent on income, consumption proportional with income, consumption proportional with the square of income, etc.), and the additive character of consumption is a general motivation for a polynomial consumption function with positive coefficients:

(1) $$C(Y) = C_0 + C_1 Y + C_2 Y^2 + \ldots + C_n Y^n \qquad C_i \geq 0 \ .$$

The condition that the income Y rises proportional with respect to the excess aggregate demand D - Y is expressed by the differential equation:

(2) $$\dot{Y} = k\left[C(Y) + I + G - Y\right]$$

in which k is a positive constant and C(Y) is given by (1).

For a constant consumption formula:

(3) $$C(Y) = c_0$$

the differential equation (2) admits the constant solution $\bar{Y}_0$ defined by

(4) $$\bar{Y}_0 = c_0 + I + G.$$

An arbitrary solution Y(t) of the equation (2) corresponding to the initial condition $Y_0 \geq 0; Y(0) = Y_0$ is given by:

(5) $$Y(t) = \bar{Y}_0 + \left(Y_0 - \bar{Y}_0\right)e^{-kt}$$

it follows that the constant solution $\bar{Y}_0$ (the equilibrium) is asymptotically stable and globally attractive; i.e. for any $Y_0 = Y(0)$, $Y(t)$ tends to $\bar{Y}_0$ in this case. The difference between aggregate demand and income D - Y (t) tends to 0.

For a linear consumption formula given by

(6) $$C(Y) = c_0 + c_1 Y; \qquad c_1 > 0$$

if $c_1 \neq 1$, then the differential equation (2) has a constant solution $\bar{Y}_1$ given by

(7) $$\bar{Y}_1 = \frac{c_0 + I + G}{1 - c_1}.$$

$\bar{Y}_1$ represents income only if it is nonnegative, $\bar{Y}_1 \geq 0$. This involves that 1- c_1 > 0, i.e. 0 < c_1 < 1. Assuming that 0 < c_1 < 1 we consider an arbitrary solution Y(t) of equation (2) corresponding to the initial condition $Y_0 = Y(0) \geq 0.$ This solution is given by

(8) $$Y(t) = \bar{Y}_1 + \left(Y_0 - \bar{Y}_1\right)e^{-k(c_1-1)t}$$

it follows that the constant solution $\bar{Y}_1$ (the equilibrium) is asymptotically stable and globally attractive, i.e. $Y(t)$ tends to $\bar{Y}_1$ and $C(Y)$ tends to $c_0 + c_1\bar{Y}_1$. In this case the difference between aggregate demand and income D - Y(t) tends to 0.

For a second degree consumption formula given by

(9) $$C(Y) = c_0 + c_1 Y + c_2 Y^2, \qquad c_2 \geq 0.$$

the differential equation (2) has constant positive solution if and only if the following two condition are satisfied:

(10) $$\begin{cases} 0 \leq \mathrm{c}_1 < 1 \\ (c_1 - 1)^2 - 4(c_0 + I + G)c_2 \geq 0 \end{cases}$$

If

(11) $$\begin{cases} 0 \leq \mathrm{c}_1 < 1 \\ (c_1 - 1)^2 - 4(c_0 + I + G)c_2 = 0 \end{cases}$$

then there is an unique positive solution $\bar{Y}_2$ given by the formula:

(12) $$\bar{Y}_2 = \frac{1 - c_1}{2c_2}.$$

If

(13) $$\begin{cases} 0 \leq \mathrm{c}_1 < 1 \\ (c_1 - 1)^2 - 4(c_0 + I + G)c_2 > 0 \end{cases}$$

then there are two positive constant solutions $\bar{Y}_{21}$ and $\bar{Y}_{22}$ given by the formulas:

(14) $$\begin{cases} \bar{Y}_{21} = \dfrac{1 - c_1 - \sqrt{(c_1 - 1)^2 - 4c_2(c_0 + I + G)}}{2c_2} \\ \\ \bar{Y}_{22} = \dfrac{1 - c_1 + \sqrt{(c_1 - 1)^2 - 4c_2(c_0 + I + G)}}{2c_2} \end{cases}$$

In the case (11) we have only one stationary solution given by the formula (12), and a solution Y(t) of the differential equation (2) corresponding to the initial condition Y_0 has the following asymptotically behavior: if $0 \le Y_0 < \bar{Y}_2$ then for $t \to +\infty$, $Y(t) \to \bar{Y}_2$; if $\bar{Y}_2 < Y_0$ then for $t \to +\infty$, $Y(t) \to +\infty$. Therefore for Y_0, satisfying $0 \le Y_0 < \bar{Y}_2$ the income Y(t) tends to $\bar{Y}_2$ and consumption tends to $c_0 + c_1 \bar{Y}_2 + c_2 \bar{Y}_2^{\,2}$. The difference D - Y(t) tends to zero. If Y_0 satisfies $\bar{Y}_2 < Y_0$ then $Y(t) \to +\infty$ and $C(t) \to +\infty$. The difference D - Y(t) $\to +\infty$ and this is not desirable.

In the case (13) we have two steady state solutions given by formula (14). A solution Y(t) of (2) corresponding to an initial condition Y_0 has the following asymptotic behavior: if $0 < Y_0 < \bar{Y}_2$ then for $t \to +\infty$, $Y(t) \to \bar{Y}_{21}$; if $Y_0 > \bar{Y}_{22}$ then for t $\to +\infty$, Y(t) $\to +\infty$. Therefore if the initial values of the income Y_0 satisfies $0 < Y_0 < \bar{Y}_{22}$, then Y(t) tends to the steady state $\bar{Y}_{21}$, and the consumption $C(t)$ tends to the constant consumption $c_0 + c_1 \bar{Y}_{21} + c_2 \bar{Y}_{21}^{\,2}$. The excess aggregate demand D - Y(t) tends to zero. If the initial values Y_0 of the income satisfies $Y_0 > \bar{Y}_{22}$ then the income Y(t) tends to $+\infty$, the consumption tends to $+\infty$ and the excess aggregate demand tends to $+\infty$.

This analysis of the stability can be made for polynomial consumption formulas of any degree. We stop here because the phenomenology suggests a possibility of the income and consumption control, choosing one or another consumption formula. We mention that the choose of a consumption formula can be considered as a modification of the coefficients in the second-degree consumption formulas.

2. Controllability of the income and consumption growth in a Keynesian macroeconomic model

We consider a Keynesian macroeconomic model in which the consumption formula is a second degree function; C(Y) given by (9) and the coefficients c_0, c_1, c_2 not satisfy the conditions (10). For fixing the ideas we assume that we have

(15) $$\begin{cases} c_1 \geq 1 \\ (c_1 - 1)^2 - 4(c_0 + I + G)c_2 = 0 \end{cases}$$

In these conditions the differential equation (2) has a stationary solution $\bar{Y}_2$ given by (12). This solution is negative and from this reason it is an income. A solution Y(t) of (2) corresponding to an initial condition $Y_0 > 0$, tends to $+\infty$ for $t \to +\infty$. This means that, in the above conditions, income Y(t), consumption C(Y(t)) and excess aggregate demand D - Y(t) tends $+\infty$ for $t \to +\infty$. To stop this tendency of excess aggregate demand an intervention in the model is suitable. This intervention can be made changing at a certain moment the coefficients of the consumption formula.

We shall illustrate this

At the moment t_1, we choose $c_1^{'}$ such that $0 < c_1^{'} < 1$ and a new formula for consumption:

(16) $$C^{'}(Y) = c_0 + c_1^{'}Y + c_2^{'}Y^2$$

in which $c_2^{'}$ is given by

(17) $$c_2^{'} = \frac{\left(1 - c_1^{'}\right)^2}{4\left(c_0 + I + G\right)}$$

After t_1 the income $Y^{'}(t)$ tends to the constant value $\bar{Y}_2^{'}$ given by

(18) $$\bar{Y}_2^{'} = \frac{\left(1 - c_1^{'}\right)^2}{2c_2^{'}}$$

and the consumption tends to the constant value given by

(19) $$C^{'}\left(\bar{Y}_2^{'}\right) = c_0 + c_1^{'}\,\bar{Y}_2^{'} + c_2^{'}\,\bar{Y}_2^{'2}$$

the excess aggregate demand D - $Y^{'}(t)$ tends to zero.

In order to demonstrate the above affirmations we shall consider the value of the income Y(t_1) at the moment t_1. Obviously Y(t_1) is a big number because Y(t) tends to $+\infty$ for t $\rightarrow$ $+\infty$. Further we consider $c_1^{'}$ satisfying:

(20) $$\max\left\{0,\ 1 - \frac{2\left(c_0 + I + G\right)}{Y\left(t_1\right)}\right\} < c_1^{'} < 1.$$

Because Y(t_1) is a big number, the ratio $\frac{2\left(c_0 + I + G\right)}{Y\left(t_1\right)}$ is less as 1 and therefore 1 - $\frac{2\left(c_0 + I + G\right)}{Y\left(t_1\right)}$ > 0. It follows that if t_1 is sufficiently large, we have $\max\left\{0,\ 1 - \frac{2\left(c_0 + I + G\right)}{Y\left(t_1\right)}\right\} = 1 - \frac{2\left(c_0 + I + G\right)}{Y(t_1)}$. From (20), it follows that we have

(21) $$Y(t_1) < \frac{1 - c_1^{'}}{2c_2^{'}}$$

where $c_2^{'}$ is given by (17). The solution of (2), in which the consumption formula is given by (19), and the initial condition is $Y^{'}(0) = Y(t_1)$ tends to $\bar{Y}_2^{'}$, the new consumption tends to $C^{'}\left(\bar{Y}_2^{'}\right) = c_0 + c_1^{'}\,\bar{Y}_2^{'} + c_2^{'}\,\bar{Y}_2^{'2}$ and the excess aggregate demand tends to zero.

Therefore, in the above case it follows that changing the coefficients c_1, c_2 we succeed to stabilize the income level, the consumption level and the excess aggregate demand decreases.

We assume now that instead (15) we have

$$(22) \qquad \begin{cases} c_1 \geq 1 \\ (c_1 - 1)^2 - 4(c_0 + I + G)c_2 > 0 \end{cases}$$

In these conditions equation (2) has two stationary solutions $\bar{Y}_{21}$ and $\bar{Y}_{22}$ which are negatives. A solution Y(t) of (2) corresponding to an initial condition $Y_0 > 0$, tend to $+\infty$ for $t \to +\infty$. Modifying at a moment the coefficients in the consumption formula it is possible to stop this tendency. In fact we can demonstrate in this case the followings:

For every moment t_1 choosing $c_1^{'}$, $0 < c_1^{'} < 1$, and the consumption formula

$$(23) \qquad C^{'}(Y) = c_0 + c_1^{'}Y + c_2^{'}Y^2$$

in which $c_2^{'}$ is given by

$$(24) \qquad c_2^{'} = \frac{\left(1 - c_1^{'}\right)^2}{4(c_0 + I + G)}$$

the income $Y^{'}(t)$ tends to $\bar{Y}_2^{'}$ given by

$$(25) \qquad \bar{Y}_2^{'} = \frac{\left(1 - c_1^{'}\right)^2}{2c_2^{'}}.$$

Consumption tends to the constant value given by

$$(26) \qquad C^{'}\left(\bar{Y}_2^{'}\right) = c_0 + c_1^{'}\bar{Y}_2^{'} + c_2^{'}\bar{Y}_2^{'2}$$

and D - $Y^{'}(t)$ tends to zero.

The case in which coefficients c_0, c_1, c_2 from the consumption formula (9) satisfy

$$\left(c_1 - 1\right)^2 - 4(c_0 + I + G)c_2 < 0$$

leads to the limitless growth of income, consumption and excess aggregate demand. This can be stabilized in the way described above.

References:

[1] Mc. Renzie - Stability of Equilibrium and the value of Positive Excess Demand. Econometrica 28:606 - 617, July 1960.

[2] Lorenz H.W. "Optimal Economic Control and Chaotic Dynamics" in Feichtinger(ed): Optimal Control Theory and Economic Analysis 3. Elsevier Science Pub., Amsterdam, 1988.

[3] Burmeister, E. & A.R. Dobell - Mathematical Theories of Economic Growth MacMillian, London, 1970.

[4] Chiarella, C. - Elements of a Nonlinear Theory of Economic Dynamics. Lecture Notes in Economics and Mathematical Systems No. 343 Springer - Verlag, Heidelberg, 1990.